P3.12 REDUCED-RANK KALMAN FILTERS: I. APPLICATION TO AN IDEALIZED MODEL OF THE WIND-DRIVEN OCEAN CIRCULATION

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1. INTRODUCTION

Estimation of the present state of the ocean and forecasting its future evolution has recently been receiving increased attention by oceanographers. Many applications have focused on the tropical Pacific (*Cane et al.*, 1996; *Verron et al.*, 1999) where large changes in oceanic circulation and surface temperature can have significant impacts on global climate. Coastal regions have also become a focus for oceanic prediction to provide estimates of local circulation for fisheries management (*Griffin and Thompson*, 1996) and to provide warning of coastal flooding due to storm surge (*Heemink and Metzelaar*, 1995). To estimate the true ocean state, information from both observations and numerical models are combined through data assimilation. The specific approaches for assimilating data are numerous and varied (*Ghil and Malanotte-Rizzoli*, 1991). The goal of this study is to evaluate a specific type of reduced-rank Kalman filter for application to realistic ocean models.

Sequential approaches (filters) use information from observations to adjust a short-term model forecast. The corrected forecast, referred to as the analysis, is then used to initialize the ocean model to produce a forecast at the next time observations are available. With the extended Kalman filter (EKF) errors are assumed to be Gaussian and their covariances are propagated through time according to the linearized model dynamics. However, for realistic oceanographic models the dimension of the state vector may be $O(10^6)$, or larger, making the storage and propagation of the required error covariances impossible. Consequently, many simplified approaches have been proposed that attempt to capture only a subset of the error covariances (*Cane et al.*, 1996; *Cohn and Todling*, 1996; *Verron et al.*, 1999). In the present study we choose the empirical orthogonal functions (EOFs) calculated from a long integration of the ocean model without assimilation to define the subspace for representing the error covariances. Versions of the filter are presented where the covariances are either dynamically evolved or an asymptotically stationary estimate is used, similar to *Fukumori and Malanotte-Rizzoli* (1995). A new extension to the use of asymptotically stationary error covariances is also evaluated that is suitable for systems that exhibit multiple quasi-stationary flow regimes. This represents an efficient means of assimilating data for systems such as the Kuroshio current or the Gulf Stream which exhibit distinct quasi-stationary flow regimes (*Spall*, 1996; *McCalpin and Haidvogel*, 1996).

A nonlinear, quasi-geostrophic, one-layer, reduced-gravity ocean model is used in a set of assimilation experiments. The model has an idealized rectangular domain that is 2048 km in the zonal direction and 4096 km in the meridional directional with a horizontal resolution of 16 km. The forcing is a steady meridionally symmetric wind field. This simple ocean model can exhibit various oceanographically relevant features, such as quasi-periodic behavior where the circulation undergoes transitions between various quasi-stationary flow regimes (high viscosity case). Also, a qualitatively different type of behavior can be obtained in which a statistically stationary state is reached with a high amount of meso-scale eddy activity (low viscosity case).

The next section provides a description of the reduced-rank Kalman filters. The assimilation experiments and results are described in Sections 3 and 4, respectively. Finally, the conclusions are given in Section 5.

2. REDUCED-RANK KALMAN FILTERS

We take an approach similar to that of *Fukumori and Malanotte-Rizzoli* (1995) and *Cane et al.* (1996) to obtain a data assimilation scheme feasible for use with realistic ocean models. To overcome the computational difficulties of applying the full EKF, the error covariances associated with the state estimate are represented only in a reduced-dimension subspace. As a result, the corrections made to the forecast at each analysis time only span this subspace. However, the full nonlinear model is used to produce the forecasts.

The EOFs used to define the subspace are calculated from a long model run without assimilation. The retained basis functions are the columns of $\mathbf{E}_r$. The dimension of the resolved subspace is typically between

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O(10) and O(10^3) compared with the dimension of the full model state that is typically O(10^6) for realistic models, but only O(10^4) for our idealized model.

To obtain the reduced-rank filter equations, the parts of the EKF that involve the error covariance matrices are projected into the EOF subspace (denoted by the subscript r):

$$\mathbf{K}_r = \mathbf{P}\mathbf{H}_r^T \left(\mathbf{H}_r\mathbf{P}_r\mathbf{H}_r^T + \mathbf{R}\right)^{-1} \quad (1)$$

$$\mathbf{P}_r^a = \left(\mathbf{I} - \mathbf{K}_r\mathbf{H}_r\right)\mathbf{P}_r \quad (2)$$

$$\mathbf{P}_r(t+1) = \mathbf{M}_r(t)\mathbf{P}_r^a(t)\mathbf{M}_r(t)^T + \mathbf{Q}_r, \quad (3)$$

where the matrix $\mathbf{H}$ is the observation operator and $\mathbf{H}_r = \mathbf{H}\mathbf{E}_r$, $\mathbf{P}$ and $\mathbf{P}^a$ are the forecast and analysis error covariances, $\mathbf{Q}$ is the model error covariances, $\mathbf{R}$ is the observation error covariances and $\mathbf{M}$ is the linearized model dynamics. The matrix $\mathbf{K}_r$ is the Kalman gain matrix that allows corrections to the forecasts to be calculated in the reduced-dimension subspace. The effect of the neglected error covariances on the resolved subspace must be approximately accounted for by modifying the observation and model error covariances.

In the case of linear dynamics, a stationary observing network, and stationary observation and model error covariances, the forecast error statistics will reach an asymptotically stationary result if all neutrally stable and unstable modes are observable. Conversely, with nonlinear dynamics (as in the present case) the error covariances will grow and decay depending on the constantly changing dynamical stability of the model. We implemented filters using stationary error covariances to obtain a highly efficient assimilation system in addition to a filter with flow-dependent covariances. To compute the stationary error covariances the ocean model was linearized with respect to the time-mean state from a long model run without assimilation. For the high viscosity model configuration, a separate Kalman gain matrix was also calculated for each flow regime by linearizing the dynamics about the mean state from each regime.

3. ASSIMILATION EXPERIMENTS

The assimilation experiments carried out for this study only simulate an application of data assimilation with real data, using the identical twin approach. Starting from a fully spun-up state, an integration of 73 years was first produced for each model configuration. This represents the "model" ocean from which the initial guess for the initial conditions were taken and the leading EOFs were calculated (using the state sampled every 88 days). Separate "true" ocean runs of 12 years for the high viscosity case and 4.7 years for the low viscosity case were also obtained starting from the final state of the corresponding "model" ocean. A

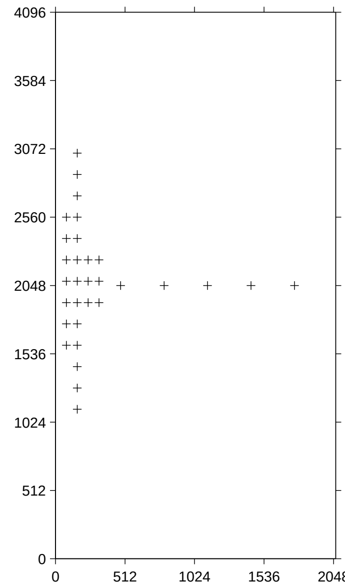

Figure 1: The 30 location where potential vorticity is observed for the assimilation experiments.

series of identical twin assimilation experiments were performed using a set of perfect observations of potential vorticity taken from the "true" ocean at 30 model grid-points distributed mostly near the western boundary (Figure 1). These observations are available every 44 days for the high viscosity case and every 9 days for the low viscosity case.

Data assimilation experiments were conducted using both flow-dependent (FD experiments) and stationary error covariances obtained with the doubling algorithm (DB experiments) in the reduced-dimension subspace. For the high viscosity case, experiments with 20 and 50 EOFs were performed. For the low viscosity case, both 50 and 100 EOFs were used. For the flow-dependent covariances, the covariances were propagated in the EOF subspace by the linearized dynamics recalculated at each analysis time with respect to the current analyzed state. Another experiment for the high viscosity case used a separate set of covariances calculated using the linearized dynamics appropriate for each flow regime (BIN experiment). The first two principle components were used to distinguish between the regimes. In theory, this approach could be expanded to employ more than three gain matrices with the model state space being partitioned using more sophisticated criteria based on several principle components or other diagnostic criteria.

A simple assimilation method, similar to optimal interpolation, was also used. For these experiments the forecast error covariances were specified to have uniform variance and Gaussian spatial correlations with

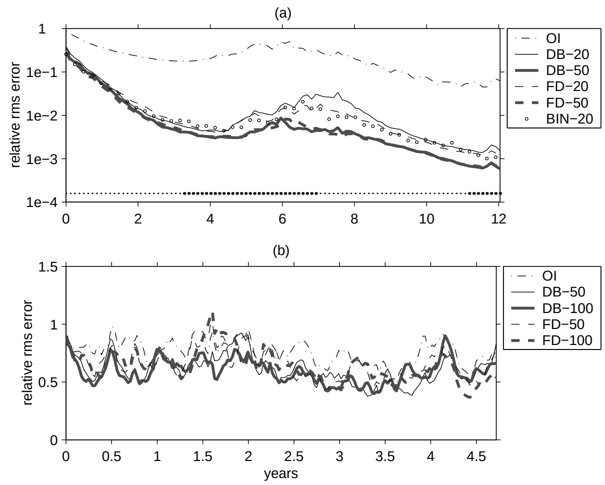

Figure 2: Evolution of normalized analysis error from all assimilation experiments. Panel (a) shows the high viscosity cases and (b) shows the low viscosity cases. The legends indicate the type of assimilation scheme used and the number of EOFs employed. The small dots near the bottom of the panel (a) indicate times when the model state is in the third flow regime and the larger dots correspond to the times of the second flow regime.

a length scale of 80 km for the high viscosity case and 48 km for the low viscosity case (OI experiments).

For comparison purposes, the model was also run from the incorrect initial conditions used to initialize the assimilation runs. The differences between the state vectors from this model run (the FALSE experiments) and the "true" ocean represent the error that would occur if no data were assimilated. These differences were used to normalize the error from all the assimilation experiments.

4. RESULTS

The rms error of the analysis from each assimilation experiment is plotted in Figure 2 after normalizing by the rms error from the run with no assimilation. Therefore a relative analysis error of 0.5 means the rms error of the run with no assimilation is reduced by half by the assimilation scheme. A relative analysis error greater than one means that by assimilating the 30 observations the analyzed state is overall more different from the "true" ocean than the state resulting from a simple model integration. All schemes for both values of viscosity were able to reduce the analysis error relative to the integration with no assimilation. The reduced-rank filters also outperformed the use of fixed Gaussian correlations for the forecast errors (OI).

For the high viscosity case (Figure 2a) all of the reduced-rank filters reduced the relative analysis error to about 1% after 2-2.5 years. However, they all suffer from an increase in error during the part of the quasi-periodic cycle when the separation point is shifted to the South (between years 4 and 7). Among the filters with 20 EOFs, the filter consisting of three separate Kalman gain matrices performs equally well as the more expensive flow-dependent filter and both reduce the analysis error greater than when a single Kalman gain matrix is used. With 50 EOFs the stationary and flow-dependent filters perform similarly, with both being consistently better than the filters with 20 EOFs. The filter with specified Gaussian correlations is only able to reduce the relative analysis error to 10% after about 9 years.

For the low viscosity case (Figure 2b) the analysis errors remains between about 45% and 90% of the error when no data are assimilated. Therefore, in this flow regime it is much more difficult to control the ocean state given the observations of potential vorticity at the same 30 locations. Both the flow-dependent and stationary reduced-rank filters are similarly effective in reducing the analysis error with the flow-dependent filter occasionally giving higher errors.

Because the error in the full state estimates can be calculated in the context of identical twin experiments, this error can be projected into the resolved and unresolved subspaces. This allows the quantification of the extent to which information from the assimilated data is transferred to the unresolved subspace through the dynamic coupling. The rms error of the analysis projected into both subspaces and normalized by the corresponding error from the run with no assimilation is shown in Figure 3. For the high viscosity case (top panels) the error in the unresolved subspace was reduced significantly overall. For the low viscosity case (bottom panels) the error in the unresolved subspace was reduced only marginally.

5. CONCLUSIONS

The reduced-rank Kalman filters evaluated in this study were able to effectively capture the dynamically relevant error covariance structure, especially for the high viscosity case, and performed much better than using covariances with a simple functional form. For the low viscosity case the reduced-rank approach reduced the analysis error only slightly more than using the simpler error covariances. It appears that more EOFs may be required to improve the reduced-rank filter for this case. The stationary filters performed surprisingly well relative to the much more computationally expensive filters with flow-dependent covariances. For the high viscosity case the use of stationary error covariances calculated separately for each distinct

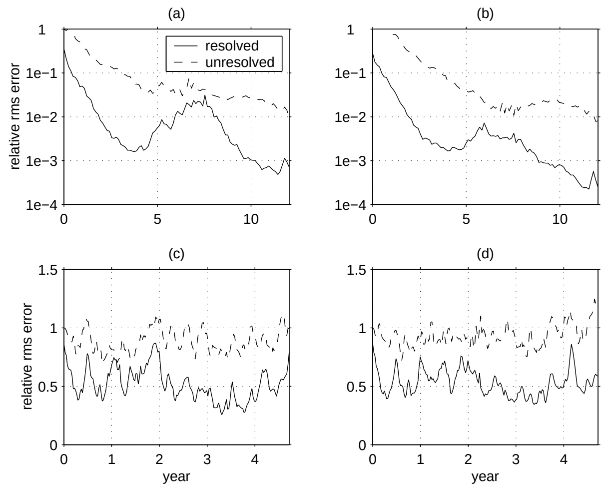

Figure 3: Evolution of normalized error in the resolved and unresolved subspaces from the assimilation experiments using the stationary reduced-rank filter. Top panels are the high viscosity cases, bottom panels are with low viscosity. Number of EOFs used: (a) 20, (b) 50, (c) 50 and (d) 100.

flow regime provided improved performance relative to using a single estimate.

The computational cost of using the reduced-rank Kalman filter with stationary error covariances is only slightly more than a simple integration of the forecast model. This is in contrast to variational and ensemble approaches that typically are O(10^2) times the cost of integrating the model.

A linearized version of the forecast model is not required for the reduced-rank Kalman filter due to the use of a numerical linearization approach. This facilitates making modifications to the assimilation system when changes are made to the forecast model code or when a completely different model is substituted. In the second part of this study this assimilation technique is applied to a more realistic three-dimensional ocean model to provide a straightforward method of assimilating altimetry data.

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