5.3 ERROR GROWTH IN UNCERTAIN SYSTEMS

Petros J. Ioannou*
Athens University, Athens, Greece

Brian F. Farrell
Harvard University, Cambridge, Massachusetts

## 1. INTRODUCTION

Linear stability theory for fluid systems has been extensively studied because of its role in advancing understanding of physical phenomena including structure and growth of perturbations, growth of errors in forecast models, transition from laminar to turbulent flow, and maintenance of the turbulent state. Historically, linear stability theory has addressed problems of deterministic growth using the method of modes (Rayleigh, 1880; Charney, 1947; Eady, 1949). However, the method of modes is incomplete for understanding perturbation growth even for autonomous systems because the non-normality of the linear operator in physical problems often produces transient development of a subset of perturbations that dominates the physically relevant growth processes (Kelvin, 1887; Farrell, 1982). Recognition of the role of non-normality in linear stability led to the development of Generalized Stability Theory (GST) (Farrell and Ioannou, 1996a). Compared to the methods of modes, the methods of GST, which are based on the non-normality of the linear operator, allow a far wider class of stability problems to be addressed including perturbation growth associated with aperiodic time dependent certain operators to which the method of modes does not apply (Farrell and Ioannou, 1996b; 1999 (FI99)). An example of such an aperiodic time dependent stability problem is the forecast error growth problem in which the non-normality of a certain but time dependent linear system, the tangent linear operator of the forecast, produces asymptotic Lyapunov instability (FI99).

There remains a class of linear stability problems still to be addressed by the methods of GST, which is the stability of uncertain systems. The problems described above involve growth of perturbations in a system with no time dependence or a system with known time dependence: the perturbations to this system may be sure or stochastically distributed and may be imposed at the initial time, or distributed continuously in time, but the operator to which the forcing is applied is considered to be certain. However, it may happen that we do not have complete knowledge of the system that is being perturbed. For instance, the parameterizations of damping and radiation in the tangent linear forecast equations are not certain but rather have a statistical variability about their mean values.

Three regimes in the statistical analysis of uncertain system are distinguished: systems in which the time dependence of the statistical fluctuations of the operator are temporally correlated for intervals short compared to the damping and oscillation time scales of the associated mean operator, systems in which the statistical fluctuations of the mean operator are correlated for time intervals long compared to the time scales of the mean operator and the physically important transitional case of operator fluctuation on time scales comparable to those of the mean operator.

## 2. EQUATIONS FOR THE ENSEMBLE MEAN

Exact dynamical equations for the evolution of an ensemble mean field under uncertain dynamics were obtained in Farrell and Ioannou, 2001a( UI). Specifically, for the uncertain linear system:

$$\frac{d\psi}{dt} = \left(A + \varepsilon\eta(t)B\right)\psi$$

where $A$ is the ensemble mean operator, $\eta(t)$ is an $O(1)$ random variable, $\varepsilon$ is the amplitude of the operator fluctuations and $B$ is the matrix of the fluctuation structure, an exact equation for the ensemble mean $\overline{\psi}(t)$ was obtained, where the bar denotes the ensemble mean over the realizations of $\eta$. For general $A$ and $B$ the equation for the ensemble mean is:

$$\frac{d\overline{\psi}}{dt} = \left(A + \varepsilon^2 BD(t)\right)\overline{\psi}\ ,$$

where:

$$D(t) = \int_0^t e^{As} B e^{-As} e^{-\nu s} ds$$

The above equations are exact for fluctuations, $\eta$, that are Gaussian, with zero mean, unit variance, and autocorrelation time $t_c = 1/\nu$.

Notable is the short autocorrelation time, $t_c << 1$, limit of the ensemble mean equation:

$$\frac{d\overline{\psi}}{dt} = \left( A + \frac{\varepsilon^2}{\nu} B^2 \right) \overline{\psi} \,.$$

This equation is generally valid when the amplitude and autocorrelation of the time fluctuations are small enough, that is, in the limit of small Kubo number, $K = \varepsilon t_c$ (Van Kampen, 1992). In particular, this equation governs the evolution of $\overline{\psi}$ for fluctuations that are temporally white (Arnold, 1992). For non-white fluctuations it is an approximate evolution equation valid for small Kubo number that will be referred to as the equivalent white noise approximation.

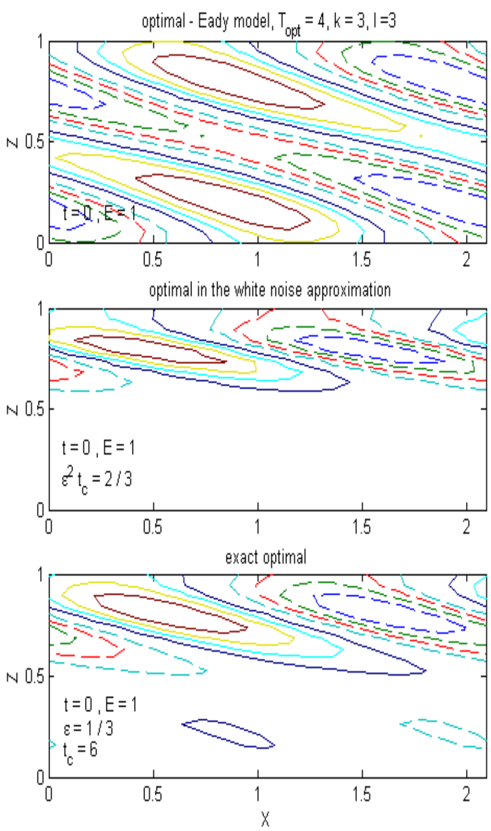


Figure 1: Optimal excitation of the uncertain Eady model with wind fluctuations concentrated near the upper boundary with form $u(z) = z^2$. The optimal perturbation is also concentrated near the upper boundary both for the exact and equivalent white noise approximate operators. This shows that the perturbation producing greatest growth for the mean operator differs greatly from the perturbation producing greatest growth when the flow fluctuations are taken into account.

Consider now the large autocorrelation limit. Let us assume that as the autocorrelation time $t_c \to \infty$ the r.m.s. amplitude of the operator fluctuation, $\varepsilon$, tends to a finite non-zero value. In such cases the propagator associated with the ensemble mean, $\overline{\Phi(t)}$, can be approximated by its quasi-static limit in which the time dependence of the operator $A + \varepsilon\eta(t)B$ is neglected but not its randomness. The ensemble mean propagator in the quasi-static approximation is consequently the average propagator over the fluctuation realizations:

$$\overline{\Phi(t)} = \overline{\exp(A + \varepsilon\eta B)}$$
$$= \int_{-\infty}^{\infty} d\eta \exp(A + \varepsilon\eta B) P(\eta)$$

where the average is taken over the probability distribution of the fluctuations, $P(\eta)$. The quasi-static approximation is formally valid for $t << t_c$. However, its validity extends for all times if all realizations of $A + \varepsilon\eta(t)B$ lead to perturbation decay, with decay times shorter than $t_c$.

### 3. OBTAINING OPTIMALS FOR ENSEMBLES

Let $\Phi(t,0)$ be the propagator associated with each realization of the operator $A + \varepsilon\eta(t)B$. In order to obtain the optimal initial condition that leads to the greatest expected perturbation growth at any future time, $t$, we can proceed in the following manner. At time $t$ the perturbation square amplitude for each realization of the fluctuations is:

$$\psi^+(t)\psi(t) = \psi^+(0)\Phi^+(t,0)\Phi(t,0)\psi(0)\,,$$

where $\psi(0)$ the initial state. It is apparent from this expression that the eigenvector of the hermitian matrix:

$$H(t) = \Phi^+(t,0)\Phi(t,0)$$

with largest eigenvalue determines the initial condition that leads to the greatest amplitude at time $t$ for that realization of the fluctuations. The other eigenvectors of $H(t)$ complete the set of mutually orthogonal initial conditions ordered according to their growth at time $t$. To obtain the expected optimal perturbation we form the ensemble average of the equation that governs the evolution of $H(t)$:

$$\frac{dH}{dt} = \left(A + \varepsilon\eta(t)B\right)^+ H + H\left(A + \varepsilon\eta(t)B\right)$$

which is satisfied by each realization. In Farrell and Ioannou (2001b) (UIb) it is shown that the ensemble mean equation of $H(t)$ is:

$$\frac{d\overline{H}}{dt} = \left(A+\varepsilon^2 BD(t)\right)^+ \overline{H} + \overline{H}\left(A+\varepsilon^2 BD(t)\right)$$
$$+\varepsilon^2\left(D^+(t)\overline{H}B + B^+\overline{H}D(t)\right),$$

where, $D(t)=\int_0^t e^{As}Be^{-As}e^{-\nu s}ds$, was previously defined in section 2 and $\overline{H}$ is the ensemble average. This equation leads directly to the equivalent white noise approximation:

$$\frac{d\overline{H}}{dt} = \left(A+\frac{\varepsilon^2}{\nu}B^2\right)^+ \overline{H} + \overline{H}\left(A+\frac{\varepsilon^2}{\nu}B^2\right)$$

$$+\frac{2\varepsilon^2}{\nu}B^+\overline{H}B$$

In their appropriate limits these equations determine $\overline{H}(t)$ at any time and eigenanalysis of $\overline{H}(t)$ in turn determines the sure optimal initial condition that leads to the largest square amplitude expected growth at time $t$. The determination of the optimal in this manner also offers constructive proof of the remarkable fact that there is a single sure initial condition that maximizes expected growth in an uncertain system.

As an example consider the Eady problem with wind fluctuations of the form $u(z)=z^2$, with r.m.s. amplitude $\varepsilon = 1/3$ and autocorrelation time $t_c = 6$. For simplicity only Rayleigh damping is included with coefficient $r=0.3$. The optimal initial conditions for optimizing the expected energy error growth at zonal and meridional wavenumbers k=l=3 and at $t=4$ (corresponding approximately 28 h) is shown in Fig. 1, while the first EOF of the evolved optimal is shown in Fig. 2. For comparison we include in the figures both the optimal for the mean operator and the optimal that is obtained using the equivalent white noise approximate operator. Note that in general fluctuations increase the expected error growth and that while the equivalent white noise approximation overestimates the growth potential, it obtains the correct structure both for the optimal perturbation and the evolved optimal covariance. Note also that optimals in the fluctuating Eady model are concentrated near the upper boundary where the fluctuations of the shear are largest.

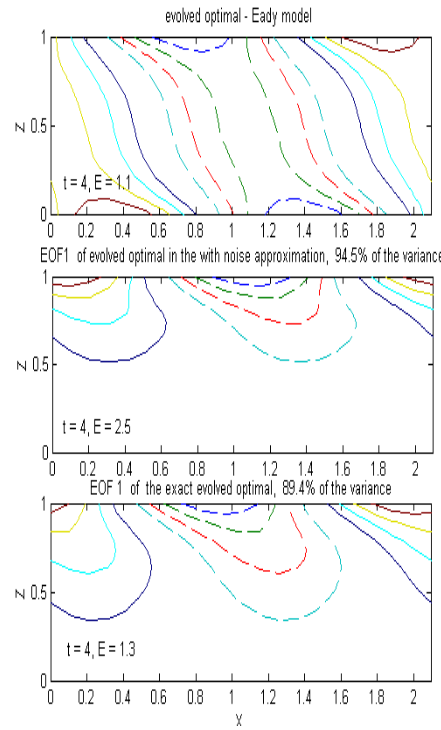


Figure 2: Evolved optimal at t=4 corresponding to the optimal initial condition in Fig. 1. The evolved optimal remains concentrated near the upper boundary showing that the mean operator does not predict the correct perturbation structure when the flow is uncertain.

## 4. CONCLUSIONS

Developing methods for analyzing perturbation dynamics in uncertain flows is important for advancing forecast theory and practice. Uncertainty may arise from a statistical description of a quantity as in a friction parameterization in a forecast model that accounts for inevitable statistical variations about the mean dissipation value or from variations arising from incomplete knowledge of the flow at one set of scales that gives rise to uncertainties at another set of scales as in the example above of the mean jet fluctuations giving rise to upper tropospheric short waves.

We have obtained and solved dynamical equations for ensemble mean quantities that are generally valid and others that are valid in the limit of both short and long autocorrelation times.

Optimal excitation plays a central role in GST and we have obtained perturbations that produce the greatest expected growth in an uncertain system and the expected structure into which this optimal evolves. Remarkably, the optimal initial perturbation is sure and moreover its structure is found in examples to differ markedly from corresponding optimal perturbations associated with the mean operator. This suggests that the choice of initial members of an ensemble to be used in operational ensemble forecast could be improved by taking forecast uncertainties into account.

This work was supported by NSF ATM-9623539 and by ONR N00014-99-1-0018.

**5. REFERENCES**

Arnold, L., 1992: *Stochastic Differential Equations: Theory and Applications*, Krieger, 228 pp.

Charney, J. G., 1947: On the dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, 4, 135-162.

Eady, E. A., 1949: Long waves and cyclone waves. *Tellus*, 1, 33-52.

Farrell, B. F., 1982: The initial growth of disturbances in a baroclinic flow. *J. Atmos. Sci.*, 39, 1663-1686.

Farrell, B. F., and P. J. Ioannou, 1996a: Generalized Stability. Part I: Autonomous Operators. , *J. Atmos. Sci,* 53, 2025-2041.

Farrell, B. F., and P. J. Ioannou, 1996b: Generalized Stability. Part II: Non-Autonomous Operators. *J. Atmos. Sci.*, 53, 2041-2053.

Farrell, B.F., and P. J. Ioannou, 1999: Perturbation Growth and Structure in Time Dependent Flows. *J. Atmos. Sci.*, 56, 3622-3639.

Farrell, B.F., and P. J. Ioannou, 2001a: Perturbation growth and structure in uncertain flows: Part I. *J. Atmos. Sci.*, (submitted).

Farrell, B.F., and P. J. Ioannou, 2001b: Perturbation growth and structure in uncertain flows: Part II. *J. Amos. Sci.*, (submitted).

Kelvin, Lord, 1887: On the stability of steady and of periodic fluid motions. *Phil. Mag.*, **23**, 459-464, 529-539.

Rayleigh, Lord, 1880: On the stability or instability of certain fluid motions. *Proc. London Math. Soc.*, 11, 57-70.

Van Kampen, N. G., 1992: *Stochastic Processes in Physics and Chemistry*, Elsevier, 465 pp.

* *Corresponding author address*: Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 15784, Greece.