8.3 QUASI-WAVELET MODELS FOR ATMOSPHERIC TURBULENCE

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1. INTRODUCTION

Turbulence is generally conceived as a collection of eddies of many different sizes (Hinze, 1975; Batchelor, 1953). The "quasi-wavelet" (QW) model discussed in this paper is an attempt to develop a mathematical representation for the turbulence that more closely resembles this physical picture than Fourier modes or customary wavelets. Like customary wavelets(Farge, 1992; Meneveau, 1994), the QW representation is based on self-similar localized functions. However, the orientations and positions of the quasi-wavelets are random, and the QW basis functions are not required to be orthonormal or to form a mathematically complete set. Some other important features of quasi-wavelets are:

- They naturally have ensemble statistics close to that of real turbulence as a consequence of the realistic basis functions.
- They can simultaneously provide information about scales of motion and spatial intermittency.
- They potentially allow simplified models of anisotropy and inhomogeneity.
- They can readily be used to generate synthetic turbulence fields.

With regard to this last point, quasi-wavelets can serve as a substitute for random Fourier modes, which have previously been used in many applications such as structural wind loading and simulation of wave scattering (Mann, 1998; Gilbert et al., 1990). In these applications, the spatially localized nature of the quasi wavelets can be advantageous (deWolfe, 1983; Goedecke and Auvermann, 1997).

This paper is organized as follows. In Section 2, we provide an overview of the formulation of the QW model. A relationship between the quasi-wavelet basis function and the energy spectrum of the turbulent velocity fluctuations is derived in Section 3. Several possible QW bases and their corresponding energy spectra are discussed in Section 4. In particular, a QW basis function that exactly yields the von Kármán spectrum is found. We also consider possible QW models corresponding to the modified von Kármán spectrum of Kristensen et al. (1989), which includes the empirical Kansas spectrum developed by Kaimal *et al* (1972) as a special case. In Section 5, we present example results.

2. OVERVIEW OF THE QW MODEL

The main goal of the QW model is to represent turbulent fluctuations by the simplest possible set of localized structures that resemble actual eddies. In the original formulation of the model (Goedecke and Auvermann, 1997), these structures were called "turbules." Here we use "quasi-wavelet," due to the localized and self-similar nature of the basis functions. For isotropic turbulence, the simplest quasi-wavelet for solenoidal velocity fluctuations was found by Goedecke and Auvermann to be a rotating spherically symmetric structure given by

$$\mathbf{v} = \nabla \times \mathbf{A}, \qquad \mathbf{A} = a^2 \mathbf{\Omega} f\left(\frac{|\mathbf{r}-\mathbf{b}|}{a}\right) \tag{1}$$

where $\mathbf{A}$ is a vector potential; a is the "size" or length scale of the quasi-wavelet, $\mathbf{b}$ is the location of its center, and $\mathbf{\Omega}$ is its angular velocity parameter. In an isotropic model, the angular velocity has a uniform, random distribution over the 4π solid angle. The scalar function f, called the QW envelope function, is any dimensionless localized function of its argument $|\mathbf{r}-\mathbf{b}|/a$. The model turbulent velocity field results from superposition of very many such quasi-wavelets. For homogeneous turbulence, each quasi-wavelet has uniformly random center location inside a chosen turbulent volume V.

Note that the individual quasi-wavelets do not necessarily satisfy the fluid equations. We merely require that an appropriate superposition of quasi-wavelets must yield the important statistical properties of the turbulence. In particular, we require in this paper that the QW model yield correct or otherwise physically reasonable spatial spectra for all ranges of the turbulent wavenumber k.

Many different sizes a_n are used in the QW model, ranging from a_1, the largest size chosen, to a_N, the smallest. Clearly, a_1 corresponds to an outer scale (length scale near the transition between the energy and inertial subranges), and a_N to an inner scale (length scale near the transition between the inertial and dissipation subranges). A fractal scaling of sizes is chosen, such that $a_2/a_1 = a_3/a_2 = \ldots = a_N/a_{N-1} =$ const. The magnitude Ω of the angular velocity $\mathbf{\Omega}$ scales with

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size a such that the characteristic speed $v = \Omega a =$ (const.)$(a^{1/3})$, which is the scaling derived in the Kolmogorov energy cascade model (Batchelor, 1953). As a result of these chosen scaling properties, the model predicts the Kolmogorov spectrum in a well-defined inertial subrange, for any physically reasonable choice of QW function f, if and only if the number density n of the quasi-wavelets of size a scales like $n =$ (const.)(a^{-3}), i. e., the QW packing fraction na^3 must be size invariant.

Other results in Goedecke and Auvermann (1997) include physically reasonable behavior of the predicted spectra in both the energy and viscous subranges. In fact, the QW model always yields energy spectra that go like k^4 for small k, then transition to $k^{-5/3}$ for intermediate k (the inertial subrange), and then fall off much faster than $k^{-5/3}$ for large k (the viscous subrange and beyond). This small k behavior is due to the fact that there is a maximum QW size a_1, while the drop-off at large k is due to the presence of a minimum QW size a_N.

3. QUASI-WAVELET THEORY OF VELOCITY SPECTRA

In Goedecke and Auvermann (1997), an expression for the velocity spectral tensor $\Phi_{ij}(\mathbf{k})$ was derived from a superposition of velocity quasi-wavelets. As shown in Goedecke et al. (2002), this expression leads to the following relationship between the energy spectrum $E(k)$ and the scale-invariant, dimensionless QW spectral function $F(y)$:

$$E(k) = \sigma_v^2 a_1 (ka_1)^{-5/3} \int_0^{ka_1} dy\; y^{14/3} \left(F(y)\right)^2 . \qquad (2)$$

Here, $\mathbf{k}$ is the wave vector, σ_v^2 is the variance of one of the velocity components, and $y = ka$, so that the integral extends over all eddy sizes a. The lower limit on the integral is actually ka_N, where a_N is the smallest scale length used. We put $a_N \to 0$ here; this can only influence the behavior of $E(k)$ in the viscous subrange. The QW envelope function $f(\xi)$ follows as the three-dimensional, inverse Fourier transform of $F(y)$:

$$f(\xi) \equiv \frac{1}{(2\pi)^3} \int d^3\xi e^{i\mathbf{y}\cdot\boldsymbol{\xi}} F(y), \qquad (3)$$

where $\boldsymbol{\xi} = (\mathbf{r} - \mathbf{b})/a$ and $\xi = |\boldsymbol{\xi}|$. Also, $(\mathbf{r} - \mathbf{b})$ is the vector from the uniformly random position $\mathbf{b}$ of the center of an eddy to the general position vector $\mathbf{r}$, so that ξ is the scaled distance (in units of a) from the eddy center. Because of the spherical symmetry, we have in general from (3)

$$f(\xi) = \frac{1}{2\pi^2 \xi} \int_0^\infty dy\; y \sin(y\xi) F(y). \qquad (4)$$

Multiplying both sides of (2) by $k^{5/3}$ and differentiating with respect to k yields

$$\frac{d}{dk}\left[k^{5/3} E(k)\right] = \sigma_v^2 a_1^{1/3} (ka_1)^{14/3} F^2(ka_1). \qquad (5)$$

4. MODEL EDDIES

As mentioned earlier, any localized QW envelope $f(\xi)$ can be normalized to yield exactly the Kolmogorov $(k^{-5/3})$ energy spectrum in an inertial subrange. Only the boundaries of the inertial subrange and the behavior of the spectra outside it are sensitive to the functional form of $f(\xi)$. In this section, we investigate several possible QW envelopes.

4.1 Von Kármán eddy

The von Kármán spectrum is commonly used in turbulence modeling. Application of this spectrum to atmospheric turbulence along with appropriate parameter values is discussed by Ostashev and Wilson (2000). The equation for the energy spectrum is

$$E_{VK}(k) = \frac{C_{VK} \sigma_v^2 k^4 L_v^5}{(1 + k^2 L_v^2)^{17/6}}, \; C_{VK} = \frac{55\Gamma(5/6)}{9\sqrt{\pi}}. \qquad (6)$$

Using (5), we now find

$$F_{VK}(y) = \left[\frac{8\Gamma(23/6)}{\sqrt{\pi}\Gamma(1/3)}\right]^{1/2} \left(\frac{L_v}{a_1}\right)^{5/2} \times \left(1 + \frac{L_v^2 y^2}{a_1^2}\right)^{-23/12} . \qquad (7)$$

The von Kármán QW envelope function $f_{VK}(\xi)$ is determined by substituting (7) into (4). The integral can be found in tables, resulting in

$$f_{VK}(\xi) = \left[\frac{\Gamma(23/6)}{\Gamma(1/3)} \frac{a_1}{L_v}\right]^{1/2} \frac{2^{-11/12}}{\pi^{7/4}\Gamma(23/12)} \times \left(\frac{\xi a_1}{L_v}\right)^{5/12} K_{5/12}\left(\frac{\xi a_1}{L_v}\right). \qquad (8)$$

where K_ν is the modified Bessel function of the second kind. The value of a_1 is flexible, although since a_1 and L_v are both outer scales, we must have $L_v/a_1 \sim 1$. Note that in the inertial subrange, $E(k) = (55\alpha/18)$ $\epsilon^{2/3}\, k^{-5/3}$, where ϵ is the rate of dissipation of the turbulent kinetic energy per unit mass and $\alpha = 0.52$ is an empirical constant. Matching this to the von Kármán energy spectrum in the inertial range $kL_v >> 1$ yields a relation among L_v, σ_v, and ϵ:

$$L_v = \left[\frac{2\Gamma(5/6)}{\alpha\sqrt{\pi}\Gamma(1/3)}\right]^{3/2} \frac{\sigma_v^3}{\epsilon}. \qquad (9)$$

4.2 Generalized von Kármán and Kansas spectra

Kristensen et al. (1989) proposed the following general equation for the one-dimensional longitudinal spectrum:

$$\chi_{||}(k) = \frac{\sigma_v^2 b(\mu) L_v}{\pi \left(1 + (kL_v)^{2\mu}\right)^{5/6\mu}}, \qquad (10)$$

with $b(\mu) = \pi\mu\Gamma(5/6\mu)/\Gamma(1/2\mu)\Gamma(1/3\mu)$. The parameter μ controls the sharpness of the transition between the energy and inertial subranges; $\mu = 1$ corresponds to the von Kármán spectrum. The longitudinal integral length scale for this model is $\mathcal{L} = b(\mu)L_v$. The corresponding energy spectrum is

$$E_{GVK}(k) = \frac{5\sigma_v^2 b(\mu) L_v (kL_v)^{2\mu}}{3\pi} \times \frac{[(2-2\mu) + (11/3)(kL_v)^{2\mu}]}{[1 + (kL_v)^{2\mu}]^{2+5/6\mu}}. \qquad (11)$$

If $\mu = 1$, this reduces to the von Kármán expression. For small k, the energy spectrum $E(k)$ for isotropic turbulence is expected to go like k^4 (Hinze, 1975). We note that E_{GVK} goes like k^4 for small k only for $\mu = 1$ or 2. The empirical spectra developed by Kaimal *et al* (1972) on the basis of the 1968 Kansas experiment correspond to $\mu = 1/2$, for which E_{GVK} is proportional to k for small k. For the Kansas spectra with $\mu = 1/2$, (5) and (11) yield

$$F_{GVK}^{\mu=1/2}(y) = \sqrt{\frac{40}{27}} \frac{(L_v/a_1)}{y^{3/2}} \frac{(1 + 14L_v y/3a_1)^{1/2}}{(1 + L_v y/a_1)^{14/6}} \qquad (12)$$

as the Fourier transform of the QW function $f_{KA}(\xi)$. This F_{KA} diverges for $y \to 0$, a direct result of the unusual behavior $E_{KA} \propto k$ for small k. We have not attempted to find a closed-form expression for the "Kansas eddy" $f_{KA}(\xi)$, mainly because the unphysical behavior of F_{KA} for small y casts doubts that the Kansas spectrum can be modeled by quasi-wavelets. Perhaps this difficulty is to be expected, because empirical spectra can contain anisotropic features that are not present in the isotropic QW model.

4.3 Gaussian and Exponential Eddies

Although the von Kármán spectrum is simple, its QW envelope function $f_{VK}(\xi)$ is not. We now show that simpler QW functions can produce spectra very close to the von Kármán spectrum. First, consider the following Gaussian-type envelope, which was used in Goedecke and Auvermann (1997):

$$F_G(y) = F_G(0)e^{-y^2/4} \to f_G(\xi) = \pi^{-3/2} F_G(0) e^{-\xi^2}, \qquad (13)$$

where the second equation results from (3). Substituting (13) into (2) and requiring that the resulting spectrum agree with the von Kármán spectrum in the inertial range, we obtain

$$F_G(0) = \left[\frac{C_{VK}}{2^{11/6}\Gamma(17/6)}\right]^{1/2} \left(\frac{a_1}{L_v}\right)^{1/3} \qquad (14)$$

and

$$E_G(k) = \frac{C_{VK}\sigma_v^2}{k^{5/3}L_v^{2/3}} \frac{\gamma\left(17/6, k^2 a_1^2/2\right)}{\Gamma(17/6)}, \qquad (15)$$

where $\gamma(p, z)$ is the incomplete gamma function.

We may assign the ratio in L_v/a_1 in several reasonable ways. For example, we could require the variances of the QW and von Kármán spectra to be the same. After a lengthy analysis, this leads to the condition $L_v/a_1 = 0.8379$. Alternatively, we could require the small k behavior of the Gaussian QW and von Kármán spectra to be the same. This choice is discussed in more detail in Goedecke et al. (2002).

A second example of a simple QW envelope consists of eddies with an exponential Fourier transform (exponential FT eddies), given by

$$F_E(y) = F_E(0)e^{-y} \to f_E(\xi) = \frac{1}{\pi^2} \frac{F_E(0)}{\left(1+\xi^2\right)^2}. \qquad (16)$$

Substituting (16) into (2) and requiring that (6) and (2) agree in the inertial range, we obtain $F_E(0)$ and the energy spectrum $E_E(k)$ for these eddies:

$$F_E(0) = \left[\frac{2^{17/3}C_{VK}}{\Gamma(17/3)}\right]^{1/2} \left(\frac{a_1}{L_v}\right)^{1/3}, \qquad (17)$$

$$E_E(k) = \left[\frac{C_{VK}\sigma_v^2}{k^{5/3}L_v^{2/3}}\right] \left[\frac{\gamma\left(17/3, 2ka_1\right)}{\Gamma(17/3)}\right]. \qquad (18)$$

Setting $\sigma_E^2 = \sigma_v^2$, we find $L_v/a_1 = 0.7238$.

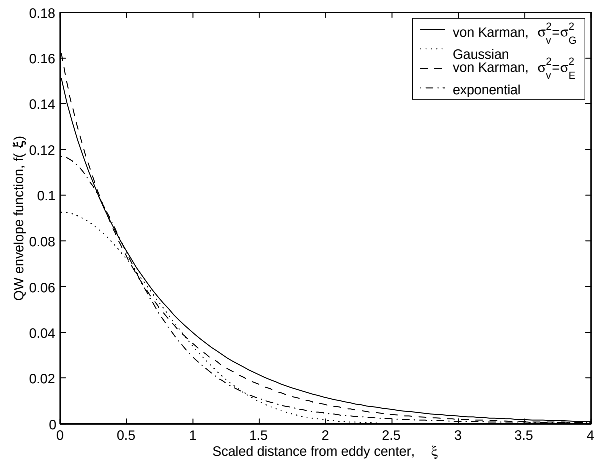

Figure 1: Comparison of various QW envelope functions.

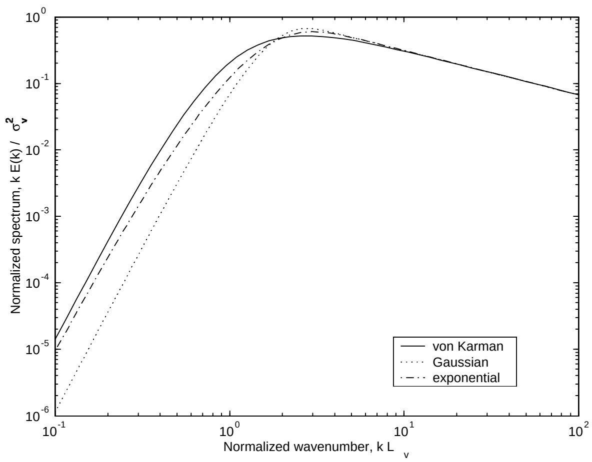

Figure 2: Comparison of energy spectra resulting from various QW envelope functions.

5. RESULTS

Various QW envelopes $f(\xi)$ derived in the preceding section are shown in Figure 1. The Gaussian and exponential envelopes are similar, both being flat at the origin. The von Kármán envelope, in contrast, has non-zero slope at the origin. Two versions of the von Kármán envelope are shown, one for variance equaling the Gaussian spectrum, and the other for variance equaling the exponential spectrum.

In Figure 2, we plot the normalized energy spectra for the von Kármán, Gaussian, and exponential QW envelopes. The spectra have been normalized to have the same variance. While they are all identical in the inertial subrange, in the energy subrange they share only the same slope (k^4). The position of the asymptote differs because the integral length scale depends on the QW envelope.

6. SUMMARY AND DISCUSSION

The von Kármán energy spectrum of turbulent velocity fluctuations has been widely used in studies of turbulence and wave (acoustic and electromagnetic) propagation in random media. In this paper, we found a QW envelope function that yields exactly the von Kármán velocity spectrum. We also showed that the QW model has flexibility extending beyond the von Kármán spectral model. In particular, it allows velocity spectra that reduce to the Kolmogorov spectrum in the inertial subrange but are adjustable in the energy subrange. This is important because models based on the von Kármán spectrum sometimes do not agree well with experimentally determined one-dimensional spectra in the energy subrange. An objective in further development of the QW model is to determine a QW function and corresponding spectrum that yields the best match to experimental data in both the energy and inertial subranges. This will require a QW model of anisotropic turbulence, which we are now attempting to formulate.

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