9.7 A NONHYDROSTATIC VERTICAL-SLICE MODEL IN A HYBRID SIGMA-THETA COORDINATE

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1. INTRODUCTION

We present the formulation of a fully compressible, nonhydrostatic hybrid ζ-coordinate model in a vertically two-dimensional (x, ζ) framework. This model is a 2D Eulerian, time-explicit analog of the 3D semi-implicit semi-Lagrangian hybrid coordinate model (Purser et al. 2002), being developed as a contribution to the multi-institution Weather Research and Forecasting (WRF) initiative.

2. MODEL FORMULATION

2.1 *Continuous equations*

There are six prognostic variables, viz., the three wind components *u*, *v*, and *w*; the potential temperature θ, the geopotential ϕ, and the modified density *m*. In flux form, the governing equations are written as:

$$\partial_t U = -\nabla\bullet(\vec{v}U) + fV - m(\partial_x M - \Pi\partial_x\theta) + \partial_\zeta \wp \partial_x \phi + mF_x , \quad (1)$$

$$\partial_t V = -\nabla\bullet(\vec{v}V) - fU + mF_y , \quad (2)$$

$$\partial_t W = -\nabla\bullet(\vec{v}W) - g\partial_\zeta \wp + mF_z , \quad (3)$$

$$\partial_t \Theta = -\nabla\bullet(\vec{v}\Theta) + mQ/\Pi , \quad (4)$$

$$\partial_t \Phi = -\nabla\bullet(\vec{v}\Phi) + gW , \quad (5)$$

$$\partial_t m = -\nabla\bullet(\vec{v}m) , \quad (6)$$

where the prognostic variables are density-weighted:

$$(U, V, W, \Theta, \Phi) = m(u, v, w, \theta, \phi) , \quad (7)$$

and the divergence term $\nabla\bullet(\vec{v}\Psi)$:

$$\nabla\bullet(\vec{v}\Psi) = \partial_x(u\Psi) + \partial_\zeta(\dot{\zeta}\Psi) , \quad (8)$$

where ψ is arbitrary. With the total air density ρ:

$$\rho = m(\partial_\zeta \phi)^{-1} , \quad (9)$$

the total pressure *p*:

$$p = p_0[(R\rho\theta)/p_0]^\gamma , \quad (10)$$

and the Exner function Π:

$$\Pi = C_p(p/p_0)^\kappa , \quad (11)$$

where p_0 is a constant reference pressure.

With the hydrostatic pressure, $\tilde{p}$, defined by

$$\partial_\zeta \tilde{p} = -m , \quad (12)$$

the nonhydrostatic-pressure deviation, $\wp$, that appear in (1) and (3) is given by

$$\wp = p - \tilde{p} . \quad (13)$$

The Montgomery stream function, $M \equiv \phi + \Pi\theta$, which appears in (1) and (3) is diagnosed from

$$\partial_\zeta M = \Pi\partial_\zeta\theta + \rho^{-1}\partial_\zeta \wp , \quad (14)$$

using vertical integration, subject to a lower boundary condition on *M*. Note in above equations, the partial derivatives wrt *x* and *t* are taken over constant ζ surfaces. The *F* terms in momentum equations denote friction and *Q* in thermodynamic equation denote the heating rate. The physical constants, *g*, *R*, C_p, $C_v = C_p - R$, $\kappa = R/C_p$ and $\gamma = C_p/C_v$ have their usual meanings.

Clearly the governing equations are valid for a generalized vertical coordinate ζ. However, to determine the *coordinate vertical velocity*, $\dot{\zeta} \equiv d\zeta/dt$, a particular definition of ζ is required. For example, ζ can be the terrain-following hydrostatic-pressure based vertical coordinate, σ:

$$\sigma = (\tilde{p}_* - \tilde{p})/(\tilde{p}_* - p_T) , \quad (15)$$

where $\tilde{p}_*$ is value of $\tilde{p}$ at the surface and p_T denotes a constant pressure at the model top. Equation (15) shows that σ varies from 0 at surface to 1 at the model top.

The hybrid σ-θ coordinate employed in this model has been proposed recently by Purser and Iredell (2002). The *generalized hybrid coordinate* is defined by

$$\zeta = \left(\sigma\frac{\vartheta}{\vartheta_T}\right)/[\sigma + (1-\alpha)\tau(\vartheta_T - \vartheta)] , \quad (16)$$

where

$$\vartheta = (1-\alpha)\hat{\theta} + \alpha(\hat{p} - \hat{p}_*) ,$$

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$$\vartheta_T = 1 - \alpha\hat{p}_* ,$$

$$\hat{\theta} = (\theta - \theta_L)/(\theta_T - \theta_L) ,$$

$$\hat{p} = (p_L - \hat{p})/(p_L - p_T) . \tag{17}$$

Here, p_L is a constant standard pressure that is somewhat larger than any typical pressure at surface or at sea-level. Similarly, θ_L is a constant potential temperature somewhat smaller than any actual value of θ encountered within the model domain, and θ_T is a specified constant value of θ at model top. There are two constant parameters α and τ. When $\alpha = 1$, we recover the σ-coordinate defined by (15).

When $\alpha = 0$, we obtain the *hybrid* σ–θ *coordinate*:

$$\zeta = (\sigma\hat{\theta})/[\sigma + \tau(1 - \hat{\theta})] . \tag{18}$$

Thus, $\alpha \in (0, 1)$ is a parameter that optionally *dilutes* the hybrid σ–θ coordinate (18) to the σ-coordinate (15). The *transition parameter* τ controls the vertical transition from a σ-like behavior low down to a more nearly isentropic behavior at upper levels.

The coordinate vertical velocity corresponding to (16) is given by

$$\dot{\zeta} = \frac{AQ}{\Pi(\theta_T - \theta_L)} + \frac{B}{p_L - p_T}\left[\int_\zeta^1 (\partial_x U)d\zeta - u\partial_x\tilde{p}\right]$$

$$+ \frac{C}{p_L - p_T}\left[\int_0^1 (\partial_x U)d\zeta - u\partial_x\tilde{p}_*\right], \tag{19}$$

where definitions of the coefficients A, B and C are omitted for brevity.

2.2 *Spatial and temporal discretization*

The model grid is unstaggered with the prognostic variables placed at the grid center; except $\dot{\zeta}$ is vertically staggered. High-order compact difference operators are employed to perform midpoint interpolation, spatial differentiation and quadrature/integral of any model variable placed at the grid centers. Computation of the spatial derivatives can be unstaggered or staggered, the latter being accomplished as a quadrature-inversion process. Spatial discretization of the model equations is entirely based on such compact difference operators (Navon and deVilliers 1987; Purser 1998).

For time discretization, an explicit two-time-level low-storage 3rd-order Runge-Kutta scheme (Williamson 1980) has been used. Moreover, to suppress grid-scale noise and to ensure nonlinear computational stability, a scale-selective spatial filter (Purser 1987) have been applied in x and ζ directions, at each time step (not each cycle of the Runge-Kutta scheme); such filters essentially damp out the grid-scale structures from the solution and are applied only to the deviations of the prognostic variables from their zonal mean or initial reference states. We have found that the pressure-deviation variable, $\wp$, also needs to be filtered during each cycle of the Runge-Kutta steps.

3. PRELIMINARY RESULTS

We have performed a number of simple tests using the model; examples include the so-called cold and warm bubble experiments. The results for the particular case of $\alpha = 1$, when the vertical coordinate ζ reduces to σ will be presented first. Then, we will present some results of the hybrid σ-θ coordinate model. The domain is cyclic in x for all experiments and the 4th-order compact schemes have been used in both x and ζ. All results are displayed using hydrostatic pressure as the vertical coordinate.

3.1 *The Cold bubble experiment*

The reference atmosphere is resting, hydrostatic and isentropic with a constant potential temperature of 300 K. A localized *temperature deficit* of the form:

$$\Delta T = -\Delta T_0 \cos^2[(\pi\beta)/2] , \tag{20}$$

only if $\beta \le 1$ where

$$\beta = \left[\left(\frac{x - x_c}{x_r}\right)^2 + \left(\frac{z - z_c}{z_r}\right)^2\right]^{1/2} , \tag{21}$$

is used to initiate the flow. The model domain is 40 km wide in x and ranges from 1000 hPa at surface to 135 hPa at top. The number of grid intervals in x is 400, with a uniform resolution of 100 m. There are 136 vertical levels that are placed on a stretched grid leading to a variable resolution that is roughly 100 m. The time step is 0.1 s, that is dictated by the CFL restriction on the vertically propagating acoustic modes in the model. Assigned values of constant parameters in (20) and (21) are $\Delta T_0 = 15°C$, $x_c = 20$ km, $z_c = 3$ km, $x_r = 4$ km, and $z_r = 2$ km. The model is run for 15 min and the results are shown for the $\zeta \equiv \sigma$ case, in a window of (20:36) km and (1000:500) hPa. Fig. 1 shows the perturbation potential temperature at 0, 300, 600, and 900 s. These results show reasonably good agreement, both qualitative and quantitative, compared to the reference solution for the cold-bubble test carried out by Straka et al. (1993).

3.2 *The Warm bubble experiment*

The reference atmosphere is isentropic as in the cold bubble test, but an initial *potential temperature excess* of the form:

$$\Delta\theta = \Delta\theta_0 \cos^2[(\pi\beta)/2] \quad (22)$$

is used to initiate the flow. Here β is defined by (20). The model domain is now 20 km wide in x and $p_T = 50$ hPa. The resolution in x is 100 m and the number of vertical levels is 136. Assigned values of the constant parameters in (22) and (21) are $\Delta\theta_0 = 7°C$, $x_c = 10$ km, $z_c = 2.75$ km $x_r = z_r = 2.5$ km. The time step is 0.1 s. The model is run for 15 min and the results are shown for the $\zeta \equiv \sigma$ case in a window of (0:20) km and (1000:100) hPa. Fig 2. shows the perturbation of θ at 540 and 720 s. These results agree reasonably well with relatively high-resolution warm-bubble runs performed by Carpenter et al. (1990) and Mendez-Nunez and Carroll (1994) using their models.

Lastly, the warm bubble experiment is performed in a realistic reference atmosphere given by the vertical profile of θ shown in Fig 3. The hybrid coordinate (16) with $\alpha = 0.1$ and $\tau = 0.2$ is used. The domain is 40 km wide in x, with $p_T = 50$ hPa and $\theta_T = 550$ K. There are 34 levels in the vertical, grid size is 1 km in x and time step is 0.1 s. The model is run for 15 min and the perturbations of u, w, and θ at 900 s are shown in Fig 3. The results show a stable integration of the model, with the hybrid vertical coordinate. We have repeated such experiments with other realistic reference states and found the model to be quite stable and accurate for a range of values of the coordinate parameters α and τ.

4. DISCUSSIONS

The 2D nonhydrostatic hybrid σ-θ coordinate model with high-order compact schemes in space and 3rd order Runge-Kutta in time is in its early development stage. Preliminary results are encouraging, but further tests are necessary to assess the strengths and weaknesses of this new hybrid coordinate model.

5. ACKNOWLEDGMENTS

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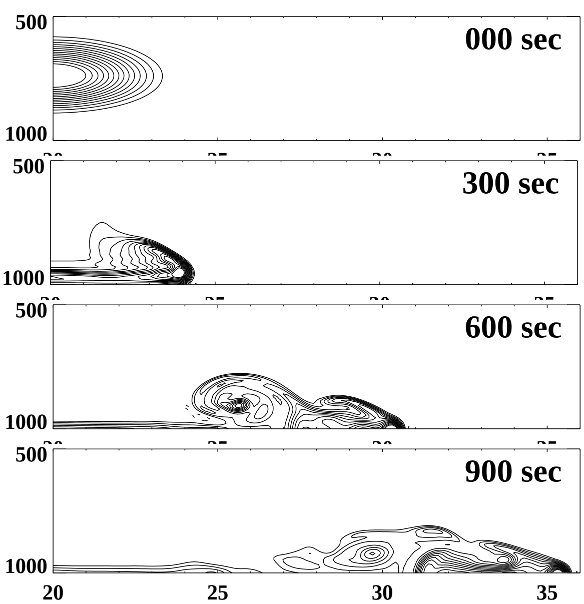


Figure 1. Perturbation of the potential temperature from the cold bubble test at 0, 300, 600, 900 s from the hybrid model

run with $\zeta \equiv \sigma$. The contour interval is 1 K. Solid contours indicate negative values. Zero contour is not shown.

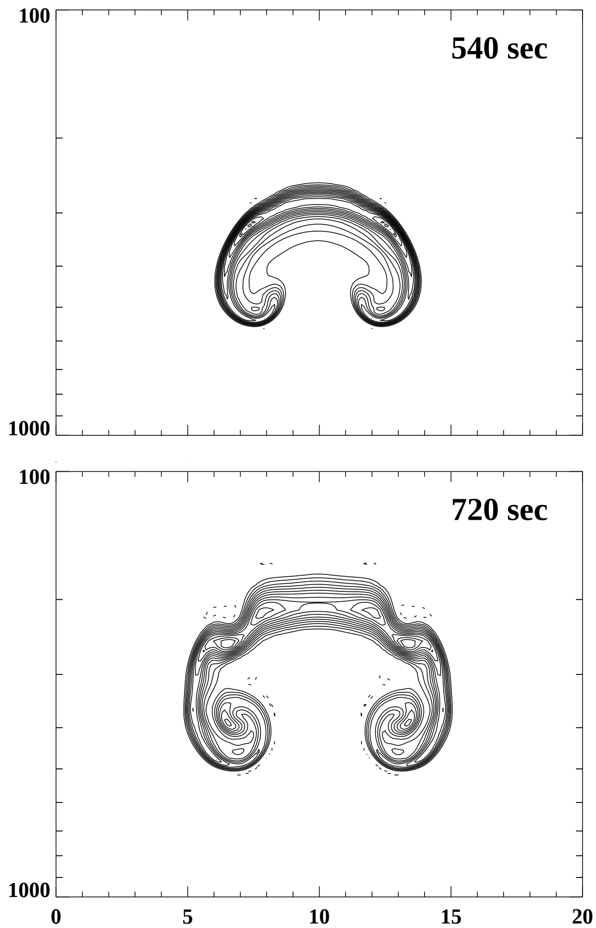
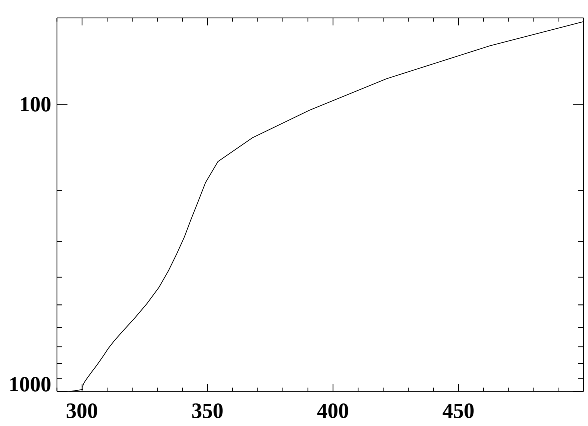


Figure 2. Perturbation of the potential temperature from the warm bubble test at 540 and 720 s of time integration of the hybrid model, with $\zeta \equiv \sigma$. The contour interval is 0.5 K. Solid contours indicate positive values. Zero contour is not shown.



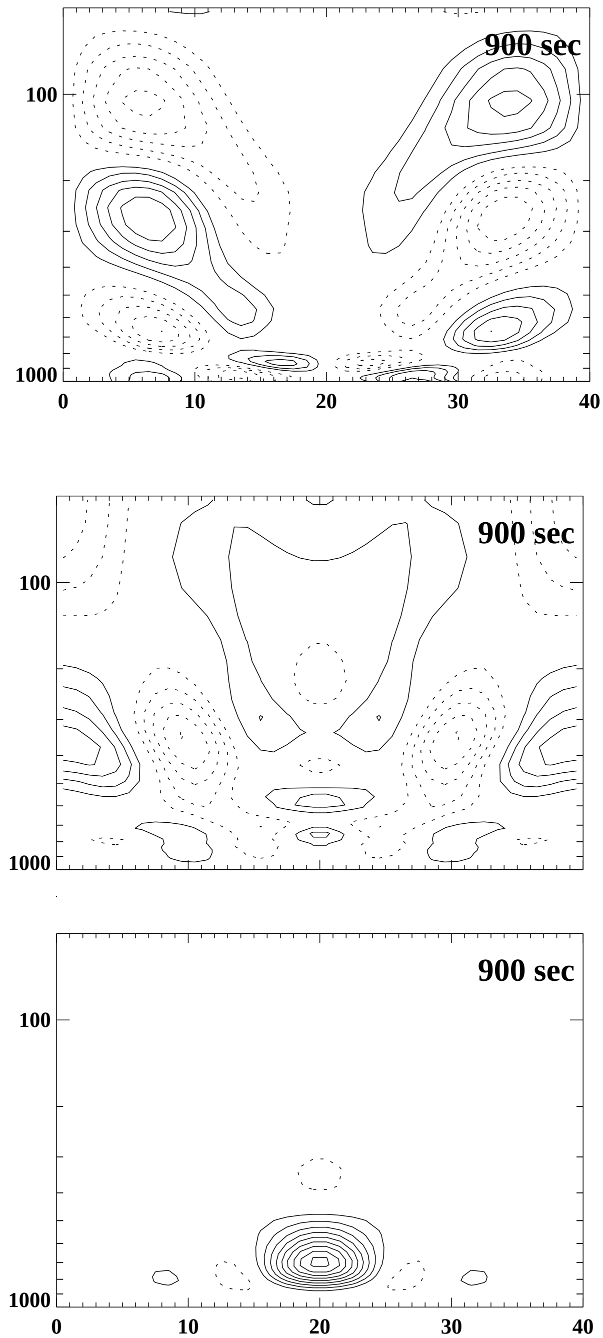


Figure 3. The 1st panel shows the vertical profile of initial potential temperature (K). The 2nd, 3rd and 4th panels show the perturbation of *u*, *w*, and θ, respectively, from the warm bubble test at 900 s from the hybrid model run with α = 0.1 and τ = 0.2. The contour interval is 0.25 ms^{-1} for the perturbations of *u* and *w*, and 0.25 K for the perturbation of θ. Solid contours indicate positive values. Zero contour is not shown.