9.1 A SEMI-LAGRANGIAN NON-HYDROSTATIC MODEL EMPLOYING A HYBRID VERTICAL COORDINATE

R. James Purser*, Sajal K. Kar†, Sundararaman Gopalakrishnan*

NOAA/NCEP/EMC
Camp Springs, MD

Tsukasa Fujita

Japanese Meteorological Agency
Tokyo, Japan

1. INTRODUCTION

We are developing a non-hydrostatic semi-Lagrangian dynamical core as a contribution to the multi-institution Weather Research and Forecasting (WRF) initiative (for example, Michalakes et al. 2000, or the web-site, http://wrf-model.org). This version is intended to complement the Eulerian dynamical core, developed at NCAR (Klemp et al. 2001).

A general description of this model is provided in Purser et al. (2001). The model variables are held on a fully unstaggered grid, which is especially convenient in a semi-Lagrangian model as it avoids the need for either multiple families of trajectories or the additional interpolations that a staggering of the variables would imply.

The numerical deficiencies of the unstaggered arrangement are largely overcome by the adoption of high-order spatial numerics (Purser and Leslie 1988). The horizontal and vertical finite differencing operators are of the 'compact' type (Navon and de Villiers 1987) and, in the case of the semi-Lagrangian advection, these operators are exploited as described in Leslie and Purser (1995) to ensure formal conservation of mass and advected scalars. The grid-to-grid semi-Lagrangian interpolations are performed using the so-called 'cascade' method of dimensional splitting (Purser and Leslie 1991). This facilitates the efficient use of forward (downstream) trajectories (Purser and Leslie 1994) which, in turn, allows a greater choice of time integration methods.

Given that semi-Lagrangian models tend to possess longer timesteps than purely Eulerian models allow, the control of time truncation errors becomes a subject of potentially greater concern. The leapfrog method, furnished with the usual time filter (Robert 1966; Asselin 1972) may therefore be a less attractive choice for a time integration scheme than some of the higher-order alternatives that a forward-trajectory semi-Lagrangian model can exploit. We are experimenting with the third-order Runge-Kutta method proposed by Williamson (1980) which can be modified to form a semi-implicit scheme as suggested by Purser (2001).

The formulation of the model is not constrained to any single particular vertical coordinate; we therefore have the opportunity to investigate whether hybrid sigma-theta coordinates, which are convincingly shown to be of benefit in relatively coarse-scale models (Uccellini et al. 1979; Johnson and Uccellini 1983; Bleck and Benjamin 1993; Johnson et al. 1993) have corresponding practical benefits at the resolution scales of between 1–10 km intended for the WRF model.

The emphasis of the present paper is the vertical 'hybrid' discretization and its accommodation within a semi-implicit modification of the low-storage Runge-Kutta scheme of Williamson (1980). Experiments with an Eulerian 'vertical slice' version of our model (Kar and Purser 2002) have been of great value in rapidly identifying strengths and weaknesses of various numerical options, including those relating to the choice of vertical coordinate. Some of these experiments employed a form of the hybrid coordinate described by Purser and Iredell (2002) which attains a constant potential temperature, θ, at the (finite) model top. This boundary condition must be supplemented by another condition on the pressure, p.

The simplest choice is to set pressure to a constant at the top, which is acceptable in idealized simulations, even for extended integrations, if this top is at a sufficient altitude (eg. Konor and Arakawa 1997). However, when it comes to forecasting with a model whose top is at a relatively modest altitude, the imposition of a constant pressure at the top isentrope would necessitate an unacceptable distortion. One solution we considered was to adopt a capping barotropic 'shallow water'

* Science Applications International Corp., Beltsville, MD; † UCAR Visiting Scientist Program

Corresponding author address: R. James Purser, W/NP2 RM 207, WWBG, 5200 Auth Road, Camp Springs, MD 20746-4304

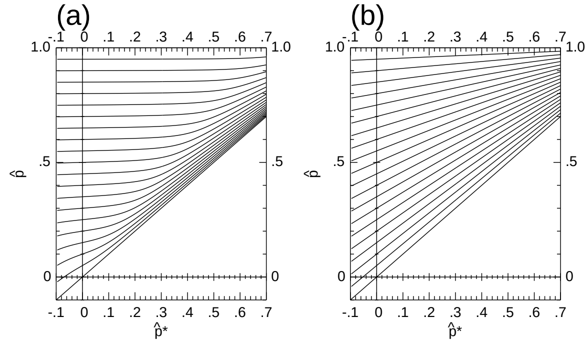

Figure 1. Two examples of the hybrid σ-p coordinate, η obtained with different combinations of the parameters β_p and τ_p. The horizontal axis shows the rescaled surface pressure $\hat{p}^*$, and the vertical axis shows the corresponding rescaled pressure, $\hat{p}$. The plotted curves identify the locations of the η coordinates at equally spaced intervals of 0.05 between 0 and 1. (a) $\beta_p = 0.1, \tau_p = 0.3$ would provide a good balance between coordinate smoothness and reducing the effects aloft of orography. (b) The choice $\tau_p = 1$, regardless of β_p, generates the modified σ-coordinate, $\eta \equiv \hat{\sigma}$.

model to supply (through its evolving depth) the non-constant hydostatic pressure condition for the top of the main domain, but this would involve considerable additional complications. Instead, following the lead of Zhu et al. (1992) we are exploring the broader class of σ-θ-p hybrid coordinates that set the top of the domain to a constant pressure without artifice. The description of this coordinate will occupy the remainder of this article.

2. HYBRID VERTICAL COORDINATES

We approach the problem of defining the final σ-θ-p hybrid coordinate, ζ, by first creating an intermediate σ-p hybrid, η, going smoothly from 0 (at the ground) to 1 at the top; then we postulate the existence of a thermodynamic function $\zeta(\eta, \theta)$ with the same range, $[0, 1]$, but whose sensitivity to θ is progressively diminished towards both the bottom and top.

In the first step, we define η implicitly by requiring a function jointly of the pressure, p, the surface pressure, p^*, and η itself, to vanish. It is convenient to linearly rescale (and reorient) the actual and surface pressures in terms of a nominal standard range, $[p_s, p_t]$, where p_t is the constant pressure intended at the model top, p_s, another constant value typical of the ground or sea level:

$$\hat{p} = \frac{p_s - p}{p_s - p_t}, \tag{1}$$

and similarly for rescaled surface pressure $\hat{p}^*$. Then a candidate for the intermediate hybrid, η, is obtained by the condition,

$$Q(p, p^*, \eta) \equiv \eta - p + g(B^+_{\beta_p} - B^-_{\beta_p}) = 0, \tag{2}$$

where,

$$g = \frac{1 - \eta}{1 - (1 - \tau_p)\eta}, \tag{3}$$

$$B^{\pm}_{\beta_p} = B_{\beta_p}(\pm\hat{p}^* - (1 - \tau_p)\eta), \tag{4}$$

and where the hyperbolic 'blending function', B_{β_p}, defined,

$$B_{\beta_p}(x) = \frac{1}{2}\left[x + (\beta^2 + x^2)\right], \tag{5}$$

asymptotes smoothly towards the line $B = 0$ for large negative x and towards $B = x$ for large positive x with a transition whose abruptness is controlled by the 'blending parameter', β_p. For the choices, $\beta_p = 0.1$, $\tau_p = 0.3$, the resulting η coordinate, plotted in the $(\hat{p}^*, \hat{p})$-plane, is shown in Fig. (1a). A smaller β_p makes the transition from p-like to terrain-following behavior sharper; a smaller τ_p causes terrain-following coordinate layers to become thinner over elevated terrain. An important special case occurs when $\tau = 1$, when the coordinate reverts to the modification of Phillip's (1957) σ-coordinate often employed when the domain's top is at a finite constant pressure:

$$\eta \rightarrow \hat{\sigma} = \frac{p^* - p}{p^* - p_t}. \tag{6}$$

This case is shown in Fig. 1b.

The coordinate of Purser and Iredell (2002), plotted as graphs in the plane of (θ, η), would typically appear as in Fig. 2a, where, in this case the top is defined as the isentrope at $\theta =$ 420 K. Limitations include the lack of safeguards against excessive thinning of the lowest layers and a breakdown of the coordinate at extremely low θ. A more important problem is the difficulty in using this coordinate for real data studies in a model with a relatively low top. A more suitable coordinate, shown in idealized form in Fig 2b, retains a condition of constant p (instead of θ) at the top, with built in limits to how thin or thick (in η) the coordinate layers can ever become, and no intrinsic restriction on the range of θ accommodated.

We have developed a formulation of the coordinate ζ in the style of Fig. 2b with parametric control over the abruptness of transitions between the adjoining regions A through E shown in this diagram. Again, we exploit hyperbolic formulae for these transitions. For example, the aymptotic values of η for a given $\zeta \in [0, 1]$ in regions A and B are given by an expression of the form,

$$\eta_{ab} = h_{ab} + g_{ab} + b_{ab}\left(\beta^2_{ab} + (\zeta - c_{ab})^2\right)^{\frac{1}{2}}, \tag{7}$$

for values of the coefficients adjusted to satisfy conditions, $\eta_{ab}(0) = 0$, $\eta_{ab}(1) = 1$, $\eta'_{ab}(0) = \tau_b$,

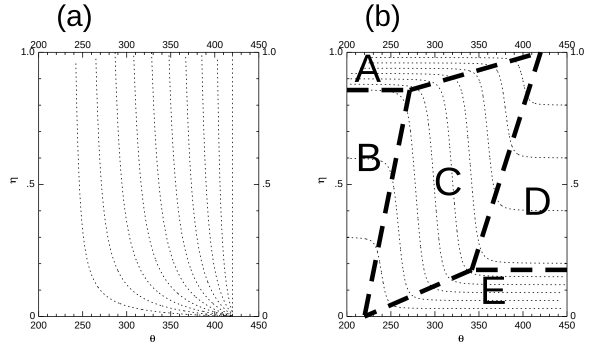

Figure 2. (a) Graphs in (θ, η)-space of coordinates ζ with a typical set of parameters obtained by the formulation of Purser and Iredell (2002). (b) Schematic depiction of an alternative formulation of the hybrid coordinate ζ that ensures that the top remains at a constant pressure and layer thicknesses at the top and bottom are regulated between predetermined thicknesses of η. The five distinct regions, *A*–*E*, are referred to in the text.

$\eta'_{ab}(1) = \tau_a$, where τ_a and τ_b are the imposed parameters controlling the extremes of layer thicknesses (in η) in the asymptotic regions of A and B of Fig 2b. With a similar construction for the asymptotes η_{de} of regions D and E, we require θ to obey,

$$\theta = \theta_s + \theta_e \eta + \theta_z \zeta + \theta_c \mathcal{P}(w), \tag{8}$$

where constant θ_s, corresponds to θ at the intersection of regions B, C and E and,

$$\theta_z = \theta_t - \theta_s - \theta_e, \tag{9}$$

for the θ_t that corresponds to the θ at the intersection of regions A, C and D. Since θ_c acts as another 'blending parameter', a convenient nondimensional alternative parameter, $\beta_c = \theta_c/\theta_z$ simplifies the prescriptive notation. The function,

$$\mathcal{P}(w) = \frac{\frac{1}{2} - w}{(w - w^2)^{\frac{1}{2}}}, \tag{10}$$

where

$$w(\eta, \zeta) = \frac{\eta - \eta_{de}(\zeta)}{\eta_{ab}(\zeta) - \eta_{de}(\zeta)}, \tag{11}$$

produces the zig-zag form of each curve of constant ζ in Fig. 2b. The degree to which the coordinate ζ mixes θ-like and p-like characteristics is under the control of the parameter, θ_e. This completes the formal definition of the proposed coordinate ζ.

In practical terms, our prescription for ζ remains implicit and it is generally easier to manipulation the algebraic formulation by inverting the functional relationships between $\mathcal{P}$ and w, and between w and η so that, by analogy with the discriminant function Q of (2), we may construct a function, $R(\eta, \theta, \zeta)$ and define the the coordinate alternatively by the condition of vanishing R, where:

$$R(\eta, \theta, \zeta) = (\eta_{ab}(\zeta) - \eta_{de}(\zeta))\, w(\mathcal{P}) + \eta_{de}(\zeta) - \eta, \tag{12}$$

where now,

$$w(\mathcal{P}) = \frac{1}{2}\left(1 - \frac{\mathcal{P}}{(\mathcal{P}^2 + 1)^{\frac{1}{2}}}\right), \tag{13}$$

and,

$$\mathcal{P}(\eta, \theta, \zeta) = \frac{\theta - \theta_s - \theta_e \eta - \theta_z \zeta}{\theta_c}. \tag{14}$$

3. DISCUSSION

The invertions of the functional relationships required to evaluate η and ζ are equivalent to constrained zero-finding problems for Q qnd R. Since all the formulae employed are smooth and easily differentiated, Newton's method is efficient, provided the first guesses are reasonably good. For initial data processing, when good first guesses for ζ are not avaliable, a combination of asymptotic approximation, based upon the reasonable idealization of function R in the regions in Fig. 2b having piecewise-linear dependence on η, θ and ζ, together with an adaptation that combines the the numerically robust (but less efficient) 'bisection method' (Conte and de Boor 1980) with Newton's method, serve to provide reliable evaluations of ζ for any realistic data profiles. In the semi-Lagrangian context, these steps can be combined cleanly with the task of locating the vertical trajectory end points. For some parameter combinations, we find that $\zeta = \eta$, so that our class of hybrids includes the original σ-p hybrid and, therefore by extension, includes also the modified $\hat{\sigma}$ coordinate.

A challenging problem in working with hybrid coordinates in the context of a semi-Lagrangian model is formulating numerically reliable solvers for the acoustic fast terms. This is not an easy problem, even in the relatively straightforward special case of $\hat{\sigma}$-coordinates, where the vertical mass distribution remains everywhere fairly equitably distributed. Preliminary explorations of this problem for the more general hybrid coordinates indicate that the vertical portion of the implied elliptic operator must be quite carefully constructed to ensure consistency with the vertical gradient operators adopted in the model. A discussion of our progress with this aspect of the model will be provided at the conference.

4. ACKNOWLEDGMENTS

This work was partially supported by the NSF/NOAA Joint Grants Program of the US

Weather Research Program. This research is also in response to requirements and funding by the Federal Aviation Administration (FAA). The views expressed are those of the authors and do not necessarily represent the official policy or position of the FAA.

REFERENCES

Asselin, R. A., 1972: Frequency filter for time integrations. *Mon. Wea. Rev.*, **100,** 487–490.

Bleck, R., and S. Benjamin, 1993: Regional weather prediction with a model combining terrain-following and isentropic coordinates. Part I. *Mon. Wea, Rev.*, **121,** 1770–1785.

Conte, S. G., and C. de Boor, 1980: *Elementary numerical analysis: an algorithmic approach. 3rd Edition.* McGraw-Hill, New York, 432 pp.

Johnson, D. R., and L. W. Uccellini, 1983: A comparison of methods for computing the sigma-coordinate pressure gradient force for flow over sloped terrain in a hybrid theta-sigma model. *Mon. Wea. Rev.*, **111,** 870–886.

——, T. H. Zapotocny, F. M. Reames, B. J. Wolf, and R. B. Pierce, 1993: A comparison of simulated precipitation by hybrid isentropic-sigma and sigma models. *Mon. Wea. Rev.*, **121,** 2088–2114.

Kar, S. K., and R. J. Purser, 2001: A non-hydrostatic vertical slice model in a hybrid sigma-theta coordinate. *(Preprint) AMS 19th Conference on Weather Analysis and Forecasting and 15th Conference on Numerical Weather Prediction*, 12th–16th August 2002, San Antonio, TX. 4 pp.

Klemp, J. B., W. C. Skamarock, and J. Dudhia, 2001: Prototypes for the WRF (Weather Research and Forecast) model. (Preprint) AMS Ninth Conference on Mesoscale Processes, Jul. 30th–Aug. 2nd 2001, Fort Lauderdale, FL.

Konor, C. S., and A. Arakawa, 1997: Design of an atmospheric model based on a generalized vertical coordinate. *Mon. Wea. Rev.*, **125,** 1649–1673.

Leslie, L. M., and R. J. Purser, 1995: Three-dimensional mass-conserving semi-Lagrangian scheme employing forward trajectories. *Mon. Wea. Rev.*, **123,** 2551–2566.

Michalakes, J., S. Chen, J. Dudhia, L. Hart, J. Klemp, J. Middlecoff, and W. Skamarock, 2000: Development of a next-generation regional weather research and forecast model. Proceedings of the Ninth ECMWF Workshop on the use of Parallel Processors in Meteorology, Reading, U.K., November 13-16, 2000. Argonne National Laboratory preprint ANL/MCS-P868-0101.

Navon, I. M., and R. de Villiers, 1987: The application of the Turkel-Zwas explicit large time-step scheme to a hemispheric barotropic model with constraint restoration. *Mon. Wea. Rev.*, **115,** 1036–1051.

Phillips, N. A., 1957: A coordinate system having some special advantages for numerical forecasting. *J. Meteor.*, **14,** 184–185.

Purser, R. J., 2001: Proposed semi-implicit adaptations of two low-storage Runge-Kutta schemes. Part I: Theoretical formulation and stability analysis. NOAA/NCEP Office Note 435, 40 pp.

——, and L. M. Leslie, 1988: A semi-implicit semi-Lagrangian finite-difference scheme using high-order spatial differencing on a nonstaggered grid. *Mon. Wea. Rev.*, **116,** 2067–2080.

——, and ——, 1991: An efficient interpolation procedure for high-order three-dimensional semi-Lagrangian models. *Mon. Wea. Rev.*, **119,** 2492–2498.

——, and ——, 1994: An efficient semi-Lagrangian scheme using third-order semi-implicit time integration and forward trajectories. *Mon. Wea. Rev.*, **122,** 745–756.

——, T. Fujita, S. K. Kar, and J. G. Michalakes, 2001: A semi-Lagrangian dynamical core for the non-hydrostatic WRF model. *(Preprint), AMS 9th Conference on Mesoscale Processes,* Fort Lauderdale, FL, 147–150.

——, and M. Iredell, 2002: A family of terrain-following theta-sigma hybrid vertical coordinates. NOAA/NCEP Office Note 436, 21 pp.

Robert, A. J., 1966: The integration of a low order spectral form of the primitive meteorological equations. *J. Meteor. Soc. Japan, Ser. 2*, **44,** 237–245.

——, 1981: A stable numerical integration scheme for the primitive meteorological equations. *Atmos. Ocean*, **19,** 35–46.

Uccellini, L. W., D. R. Johnson, and R. E. Schlesinger, 1979: An isentropic and sigma coordinate hybrid numerical model: Model development and some initial tests. *J. Atmos. Sci.*, **36**, 390–414.

Williamson, J. H., 1980: Low storage Runge-Kutta schemes. *J. Comput. Phys.*, **35,** 48–56.

Zhu, Z., J. Thuburn, B. J. Hoskins, and P. H. Haynes, 1992: A vertical finite-difference scheme based on a hybrid σ-θ-p coordinate. *Mon. Wea. Rev.*, **120**, 851–862.