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UNCERTAINTY IN NUMERICAL MESOSCALE MODELING BY THE USE OF A TOPOGRAPHY DEFINED VIA MAP PROJECTIONS

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1. Introduction

Digital elevation models are the standard way to describe the Earth surface with respect to a spherical earth model. DEM's play a basic role in the numerical solution of mesoscale model equations because they determine (i) the minimum horizontal grid spacing to describe accurately the topography in a model domain and (ii) the $\sigma-$type vertical coordinates, which have shown to be an efficient way to incorporate topographic irregularities into the model equations. Mesoscale models such as MM5 (Duhia et. al., 1999), RAMS (Pielke et. al., 1992) and ARPS (Xue et. al., 1995) use conformal map projections to represent the Earth surface on a plane surface and generate a DEM on a cartesian regularly-spaced computational grid. In this work it is shown that the this procedure has no advantage because it generates a DEM with an incorrect terrain elevation whose error can be larger than the terrain elevation datum itself, and this in turn increases the uncertainty of model equations in $\sigma-$type vertical coordinates.

2. Coordinate systems

Consider a spherical earth model with radius a. The primary cartesian coordinate system XYZ is defined with its origin at the earth's center, is fixed to the earth and the Z axis coincides with the earth's rotation axis. Let λ, ϕ, r denote the usual spherical coordinates of an air parcel with cartesian coordinates X, Y, Z; $X = r\cos\phi\cos\lambda$, $Y = r\cos\phi\sin\lambda$, $Z = r\sin\phi$. The tangent-plane coordinate system xyz has its origin at a point $(\lambda_c = 0, \phi_c, r = a)$ on the earth, the x (y) axis is tangent to the parallel circle (meridian) at (λ_c, ϕ_c), is positive eastward (northward), and the z axis is taken out of the earth. The relation between the coordinates XYZ and xyz is

$$\begin{pmatrix} x \\ y \\ z+a \end{pmatrix} = \mathbb{R}_c \begin{pmatrix} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{pmatrix} \tag{1}$$

where the matrix R_c is given by

$$\mathbb{R}_c = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\phi_c & -\sin\phi_c & \cos\phi_c \\ \cos\phi_c & 0 & \sin\phi_c \end{pmatrix}.$$

To analyze the role of map projections let us consider the formal definition of the projection coordinates $x_p y_p H_p$. Let $x_p y_p$ be a cartesian coordinate system on a projection plane P which is normal to the H_p axis. The projection of a point (λ, ϕ) on the terrestrial sphere is the point (x_p, y_p) given by a pair of projections equations

$$x_p = P_x(\lambda, \phi) \qquad y_p = P_y(\lambda, \phi). \tag{2a}$$

Usually, the center (λ_c, ϕ_c) of the horizontal model domain D on the tangent plane xy is projected on the origin of the $x_p y_p$ system, $P_x(\lambda_c, \phi_c) = P_y(\lambda_c, \phi_c) = 0$, and the eastward parallel circle and the northward meridian on (λ_c, ϕ_c) are projected on the positive x_p and y_p axes, respectively. If a point in physical space has spherical coordinates (λ, ϕ, r) the coordinate H_p is defined by

$$H_p = r - a. \tag{2b}$$

Thus we have four equivalent sets of coordinates to define the position of a parcel, namely, (x, y, z), (X, Y, Z)

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and (λ, ϕ, r) which have a simple geometrical interpretation in physical space while (x_p, y_p, H_p) are coordinates in an abstract space. From (2a,b) we get λ, ϕ, r in terms of x_p, y_p, H_p: $\lambda_p = \lambda(x_p, y_p)$, $\phi_p = \phi(x_p, y_p)$, $r_p = H_p + a$. This together with (1) yields the relation between x, y, z and x_p, y_p, H_p, namely,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbb{R}_c \begin{pmatrix} (H_p + a)\cos\phi_p \cos\lambda_p \\ (H_p + a)\cos\phi_p \sin\lambda_p \\ (H_p + a)\sin\phi_p \end{pmatrix}. \qquad (3)$$

This shows that xyz and $x_p y_p H_p$ are different coordinate systems. However, the documentation of mesoscale models such as RAMS, ARPS or MM5 do not report or suggest the use of these coordinate transformations. Instead, the approximation

$$x_p \sim x \quad y_p \sim y \quad H_p \sim z \qquad (4)$$

is used to work on the cartesian coordinate system xy with the expectation that the map projection considers the spherical shape of the earth. In fact, the usual horizontal coordinate system in mesoscale modeling is the cartesian system xyz but models such as ARPS, RAMS or MM5, attempt to consider the earth sphericity using map projections to define the topography. To analyze this approach consider the horizontal domain on the xy plane $D(L) = [-L, L] \times [-L, L]$. Three domains illustrate the magnitude of the domains used in mesoscale modelation, namely, the domains $D_a\,(L_a \sim 665\text{ km})$ and $D_b\,(L_b \sim 882\text{ km})$ which were used in the study of regional transport of atmospheric pollutants by Yamada et. el (1989), and Fast and Zhong (1998), respectively, and the domain $D_c\,(L_c \sim 1665\text{ km})$ which is used by the mexican meteorological service (SMN) for operational meteorological analysis (SMN, 2002).

3. Topography defined via map projections

The terrain elevation data are referred to an ellipsoid but we can consider that the data are known with respect to an spherical earth model defined properly from the ellipsoidal model, see, e.g., Nuñez (2002). If $h(\lambda, \phi)$ denotes the terrain elevation on the point (λ, ϕ) of the terrestrial sphere, then the set of points with spherical coordinates $(\lambda, \phi, r = h + a)$ define the true earth surface (which is called geoid). In practice the geoid is known only on a discrete set of points $(\lambda_k, \phi_k, r_k = h_k + a)$, $k = 1, .., N$, which define a DEM $\{\lambda_k, \phi_k, h_k\}$. According to the documentation of MM5, RAMS and ARPS, to define the topography with respect to the coordinate system xyz with a given DEM $\{\lambda_k, \phi_k, h_k\}$, we compute a point (x_{pk}, y_{pk}) with a map projection (4.1a),

$$x_{pk} = P_x(\lambda_k, \phi_k) \qquad y_{pk} = P_y(\lambda_k, \phi_k),$$

and we consider that the terrain elevation on (x_{pk}, y_{pk}) is h_k because map projections generate a minimum distortion of the earth surface. Of course, (x_{pk}, y_{pk}) is on the projection plane P in an abstract space $x_p y_p H_p$, but if the terrain height on the domain center (λ_c, ϕ_c) is defined as the datum $h(\lambda_c, \phi_c)$, the plane P can be identified as the xy plane tangent to the earth at (λ_c, ϕ_c) and every point (x_p, y_p) defines a point in the xy system, see Nuñez (2002) for details. According to the definition of projection coordinates, it is clear that a point P on the geoid with coordinates λ, ϕ, h, has projection coordinates $x_p, y_p, H_p = h$, spherical coordinates $\lambda, \phi, r = h + a$ and the unique and correct coordinates x, y, z of P are obtained from (3). If the projection coordinates (x_p, y_p, H_p) of P are seen as the coordinates of a point in physical space rather than in the abstract space $x_p y_p H_p$, such coordinates define the localization of point P^* different to P. Thus, in general, we have

$$x \neq x_p \qquad y \neq y_p \qquad z \neq H_p.$$

Since map projections generate a minimum distortion of the terrestrial sphere, the horizontal coordinates are very similar over a wide range, $x \sim x_p$, $y \sim y_p$. For instance, the figs. 4, 5 reported in Nuñez (2002) show that the relative error $|y - y_p|/y$ is very small for $y \in [0, 1665\text{km}]$ and several map projections. However, the problem lies in the vertical coordinate. If (x_p, y_p) is close to the origin $(x = 0, y = 0) = (\lambda_c, \phi_c)$ the difference $|z - H_p|$ is small but it increases rapidly as x_p or y_p do. Consider, for example, a zero terrain elevation $H_p = h = 0$, in this case the geoid coincides with the terrestrial sphere and therefore the exact terrain elevation $z_h\,(x, y)$ on a point (x, y) is $z_h\,(x, y) = -a + \sqrt{a^2 - x^2 - y^2}$, whereas the terrain elevation on a point (x_p, y_p) obtained via map projections is $h = 0$, and the error of the terrain elevation on the corner $(x = L, y = L)$ of the domain $D\,(L)$ is

$$|h - z_h\,(L, L)| = a - \sqrt{a^2 - 2L^2}.$$

In particular, if $a = 6378$ km the error on the corner $(x = L, y = L)$ of the domains D_a, D_b, D_c is 69 km, 123 km and 450 km, respectively. It is clear that if we consider real DEM $\{\lambda_k, \phi_k, h_k\}$ the terrain elevation error is basically the same because $|h_k| << |z_h\,(L, L)|$. Some terrain elevation data from GTOPO30 (U.S. Geological Survey, 1997) have the uncertainty $\Delta h = \pm 30$ m. In this case, it can be shown that the approximation $H_p \sim z$ is consistent with this uncertainty on a horizontal domain $D_h \sim 60 \times 60$ km^2 which is very small with respect to the domains D_a, D_b, D_c (Nuñez, 2002).

4. Bidimensional steady flow

In this section we compare mesoscale flow equations obtained from a topography defined via map projections and those from a correct representation of topography. By simplicity the earth rotation is ignored and the flow is isothermic, incompressible and inviscid so that the model equations with respect to the xyz system are

$$\nabla \cdot \mathbf{v} = 0 \qquad \frac{d\mathbf{v}}{dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = 0 \tag{5}$$

where $v = u\hat{x} + v\hat{y} + w\hat{z}$. Consider the terrain-following coordinate system $y^1 = x$, $y^2 = y$ and

$$y^3 = H[z - z_h(x, y)]/[z_{\max} - z_h(x, y)]$$

were $z_h(x, y)$ is the correct terrain elevation on the point (x, y) and H is the height of the model domain. The momentum equation in the y^i system has the form (Pielke, 1984).

$$\frac{\partial v^k}{\partial t} + v^m v^k{}_{,m} + G^{km}\frac{\partial p}{\partial y^m} + \frac{\partial y^k}{\partial z} g = 0. \tag{6}$$

If the point (λ, ϕ) on the terrestrial sphere is projected on a point (x_p, y_p) and the terrain elevation $z_{hp}(x_p, y_p)$ on this latter one is the geoid datum $h\,(\lambda, \phi)$, $z_{hp}(x_p, y_p) \equiv h\,(\lambda, \phi)$,the correct terrain-following coordinates in the abstract space $\mathrm{x}_p\mathrm{y}_p\mathrm{H}_p$ are $\tilde{y}_p^1 = x_p, \tilde{y}_p^2 = x_p$ and

$$\tilde{y}_p^3 = H_{\max}[z - z_{hp}(x_p, y_p)]/[H_{\max} - z_{hp}(x_p, y_p)].$$

However, if (x_p, y_p, H_p) are considered as the cartesian coordinates (x, y, z) of a point with spherical coordinates $(\lambda, \phi, r = H_p + a)$, we replace x_p, y_p by x, y to get the coordinates $y_p^1 = x$, $y_p^2 = x$ and

$$y_p^3 = H_{\max}[z - z_{hp}(x, y)]/[H_{\max} - z_{hp}(x, y)],$$

and the momentum equations has the form

$$\frac{\partial v_p^k}{\partial t} + v_p^m v_p^k{}_{,m} + G_p^{km}\frac{\partial p_p}{\partial y_p^m} + \frac{\partial y_p^k}{\partial z} g = 0 \tag{7}$$

where the index p denotes quantities in the y_p^i system.

Mesoscale models that use map projections to define topography in the xyz system solve equations like (7) which are approximations of equations like (6) which are obtained with the correct topography equation $z_h(x, y)$. The results of the previous section show that the error of $z_{hp}(x, y)$ can be larger than the height H used in some mesoscale studies, a result that can invalidate the usefulness of model equations like (7). In principle we have to solve equations (6) and (7) to compare the meteorological fields and determine the error generated by the inaccuracy of $z_{hp}(x, y)$. We consider the stationary flow around a two dimensional earth with radius a and topography $h\,(\lambda, \phi) = 0$ on the yz plane. The governing equations are

$$\nabla \cdot \mathbf{v} = 0 \qquad\qquad \frac{1}{\rho}\nabla p = -(\mathbf{v} \cdot \nabla)\mathbf{v} - g\hat{\mathbf{z}}.$$

The (correct) topography equation and the boundary condition for v are

$$z_h\,(x, y) = -a + \sqrt{a^2 - y^2} \quad \mathbf{v} \cdot \hat{\mathbf{n}}|_{z = z_h} = 0.$$

where the vector $\hat{n}$ is normal to the earth. The velocity field $v = v\hat{y} + w\hat{z}$ which satisfies the boundary condition and the continuity equation can be obtained from the flow around a circular cylinder and is

$$v = v_0(1 + R^{-2} - 2\bar{y}^2 R^{-4}) \quad w = -2v_0\bar{y}(1 + \bar{z})R^{-4}$$

where $R = [\bar{y}^2 + (1 + \bar{z})^2]^{1/2}$, $\bar{y} = y/a$, $\bar{z} = z/a$. The pressure field p is obtained from the momentum equation. The pressure p and the velocity v in terrain-following coordinates y^i are obtained from (6) but to analyze the fields p_p, v_p from map projections it is enough to work with p and v. If the terrain elevation on the earth is zero, $h\,(\lambda, \phi) = 0$, we have $z_{hp}\,(x, y) = 0$ and the terrain-following coordinates are $y_p^1 = x$, $y_p^2 = y$, $y_p^3 = z$ for *any* map projection. This means that the earth is approximated by the xy plane, an approximation which is valid only in a vecinity of the origin $(x = 0, y = 0, z = 0)$. If we set $v_p^2 = v_p$, $v_p^3 = w_p$ and $v_p = v_p\hat{y} + w_p\hat{z}$ the continuity and momentum equations are

$$\frac{\partial v_p}{\partial y} + \frac{\partial w_p}{\partial z} = 0 \qquad \frac{1}{\rho}\nabla p_p = -(\mathbf{v}_p \cdot \nabla)\mathbf{v}_p - g\hat{\mathbf{z}}.$$

where w_p has to satisfy the boundary condition $w_p = 0$ for $z = 0$. The solution to this problem is a uniform flow and if the condition $v = v_p$ is imposed on $y = z = 0$ then eq. (4.9) yields $v_p = v_p\hat{y} + w_p\hat{z}$ with $v_p = 2V_0$, $w_p = 0$ and $p_p = -\rho g z + const$. The pressure field p can be obtained from eqs. (5) but to simplify computations we compare the isobars which are calculated as follows. Let $z^*(y; y_0, z_0)$ and $z_p^*(y; y_0, z_0)$ be the isobars that pass through the point (y_0, z_0). If we impose the boundary condition $p = p_p$ on $(y_0 = 0, z_0)$, p and p_p have the same pressure value on the isobars $z^*(y; 0, z_0)$ and $z_p^*(y; 0, z_0)$, respectively. We have $z_p^*(y; 0, z_0) = z_0$ and z^* is obtained by solving the ordinary differential equation

$$\frac{dz^*}{dy} = \frac{\partial p}{\partial y}\left(\frac{\partial p}{\partial z}\right)^{-1}$$

with the boundary condition $z^*(0; 0, z_0) = z_0$. The following table reports the correct terrain elevation $z_h(x, y)$, velocity components v, w and the percentage error $\delta v_p = |v_p/v - 1| \times 100$ for $z_0 = 0$, $y \in$[0,1665km], $a = 6378$ km and $v_0 = 5$ ms^{-1}, with $v_p = 10$ ms^{-1} and $w_p = 0$ for all y, z,

y	z_h	v	w	δv_p
0	0	10	0	0
250	-5	10	-.4	0.2
650	-20	9.8	-1.0	1.6
882	-79	9.7	-1.3	2.9
1665	-179	9.1	-2.3	10.1

Thet δv_p increases from 0 to 10% as y goes from 0 to 1665 km while $|z_h|$ and $|w|$ increase up to 179 km and 2.52 ms^{-1}, respectively, although the isobars $z^*(Y; 0, 0)$ and $z_p^*(Y; 0, 0)$ are essentially the same. The difference between v and v_p is large for $y \in$[0,1665] and it is clear that such a difference comes from the inaccuracy of $z_{hp}(x, y)$, which in turn is generated by the earth curvature and the use of map projections. In this example we have ignored important factors controlling the fluid motion such as the tridimensional nature of the problem, the earth rotation, the stratification, a complex topography and the time evolution of atmospheric flows, but we may expect that these factors will generate larger differences between the meteorological fields obtained from the correct topography $z_h(x, y)$ and those from $z_{hp}(x, y)$. To this we have to add the known fact that in a nonstationary flow the differences between velocities v, v_p and pressures p, p_p reported in the previous tables can generate qualitatively different mesoscale flows as the time t increases because of the nonlinearity of governing equations. These differences may be particularly important for some mesoscale meteorological applications such as the study of diffusion and transport of pollutants which are phenomena that depend of an accurate description of small scale motions.

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