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DOMAIN OF VALIDITY OF SOME COMPUTATIONAL MESOSCALE MODELS

Marco A. Nuñez *

Universidad Autónoma Metropolitana Iztapalapa

1. Introduction

The fundamental momentum vector equation for an air parcel in any coordinate system fixed to the earth is

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla P + \mathbf{g} - 2\tilde{\mathbf{\Omega}} \times \mathbf{V} + \mathbf{f} \qquad (1)$$

where $\mathbf{V}$, ρ, p are the velocity vector, density and pressure, respectively, $\tilde{\mathbf{\Omega}}$ is the earth's angular velocity, $\mathbf{f}$ is a frictional-force and terms with Ω^2 are neglected, see, e.g., Dutton (1976). By simplicity we consider a dry, isothermic and inviscid atmosphere, $\mathbf{f} = \mathbf{0}$. If we consider a uniform-mass spherical earth, the gravitational acceleration is given by

$$\mathbf{g} = -g\frac{a^2}{r^3}\mathbf{R}$$

with $g \equiv GMa^{-2}$, $\mathbf{R}$ being the vector from the earth's center to the parcel, $r = \|\mathbf{R}\|$, M and a are the mass and radius of the earth and G is the gravitational constant. The usual coordinate system in the standard mesoscale literature is a cartesian system xyz with its origin at a point P_c in latitude ϕ_c on the terrestrial sphere, the xy plane is normal to $\mathbf{g}$ at P_c and the z axis is outside of the earth. It is generally acknowledged that when the horizontal scale of the motion L ($|x|, |y| \leq L$) is of order 10^3 km or smaller, the gravitational acceleration $\mathbf{g}$ can be taken as a constant and normal to the xy plane. Thus, the common form of the momentum equation used in the mesoscale literature is

$$\frac{d\mathbf{v}^0}{dt} = -\frac{1}{\rho^0}\nabla p^0 - g\hat{\mathbf{z}} - 2\tilde{\mathbf{\Omega}} \times \mathbf{v}^0, \qquad (2)$$

where $\hat{\mathbf{z}}$, $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ are the unit vectors of the xyz system, see, e.g., Pielke (1984). This is a simple equation to perform theoretical analyses with the effects of rotation included, which has been used by numerical mesoscale models to treat problems with complex topography. In this latter case the z coordinate is replaced by either a σ_z or σ_p coordinate to simplify the treatment of lower boundary conditions. Following this scheme, mesoscale models such as MM5 (Duhia et. al., 1999), RAMS (Pielke et. al., 1992), ARPS (Xue et. al., 1995), or HOTMAC (Yamada et. al., 1981), which solve the momentum equation with $\mathbf{g} \sim -g\hat{\mathbf{z}}$ and coordinates $xy\sigma_z$ or $xy\sigma_p$, have been developed. Although some authors have pointed out that the range of validity of the equation (2) may be small (see, e.g., McVittie (1948), Dutton (1976)), some applications of these models have considered an horizontal domain $\mathcal{D}(L) = 2L \times 2L$ with $L \gtrsim 650$ km, whereas the results reported below suggest that the approximation $\mathbf{g} \sim -g\hat{\mathbf{z}}$ is valid on $\mathcal{D}(L^0_{\max})$ with $L^0_{\max} \lesssim 100$ km.

Two problems motivate the use of a large horizontal domain $\mathcal{D}(L)$. The first is the necessity of including the influence of propagating synoptic disturbances on the regional weather. For instance, Pielke (1994) suggests a domain of at least 5000 km on a side to reasonably resolve some disturbances in winter. The second is that the boundary errors induced by artificial boundaries do not contaminate the results with a large $\mathcal{D}(L)$. The solution of these problems is incompatible with the small domain of validity $\mathcal{D}(L^0_{\max})$ of the numerical models that solve (2) in $xy\sigma$ coordinates. The answer to this conflict is the use of the exact $\mathbf{g}$. If a parcel is at the point (x, y, z) at time t we have $\mathbf{R} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z + a)\hat{\mathbf{z}}$ and the

*Depto. de Fisica, Univ. Autónoma Metropolitana Iztapalapa, A. P. 55-534, CP 09340, Distrito Federal, Mexico. e-mail: manp@xanum.uam.mx .

correct momentum equation is

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - g\frac{a^2}{r^3}[x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z+a)\hat{\mathbf{z}}] - 2\tilde{\mathbf{\Omega}}\times\mathbf{v}, \quad (3)$$

whose numerical implementation requires a small modification of the current mesoscale software. Although the coordinate system xyz is not well suited for practical applications to large-scale problems, the equation (1) is valid for *any* coordinate system rotating with the earth and this includes the xyz system. Thus, the eq. (3) together with the conservation equations of mass, energy, moisture and the equation of state, provides the correct meteorological fields when the correct initial and boundary conditions are used, independently of the magnitude of the domain $\mathcal{D}(L)$. This is illustrated below, where a simple problem shows that the eq. (3) can yield the correct pressure field on the *whole* earth.

Map projections have been used in atmospheric modeling with the purpose of including the earth sphericity into model equations (Haltiner, 1971). Accordingly, some mesoscale computational systems, such as MM5 and RAMS which use coordinates $xy\sigma_z$ or $xy\sigma_p$, include metric factors in the horizontal derivatives of model equations to consider map projections. This has motivated the use of such computational systems on domains $\mathcal{D}(L)$ with L as large as 885 km (Fast and Zhong, 1998) or 1665 km (Servicio Meteorológico Nacional (SMN), 2002). In principle, the use of map projections generates coordinate systems $x_p y_p z_p$ which are legitime to solve model equations. However, we show that if x_p, y_p, z_p are taken as correct approximations of x, y, z, respectively, the horizontal momentum equations omit the gravitational acceleration and, therefore, their reliability region is similar to $\mathcal{D}(L^0_{\rm max})$.

2. Analysis of the approximation $\mathbf{g} \sim -g\hat{\mathbf{z}}$

There are three domains which illustrate the magnitude of the domains used in mesoscale modelation, namely, the domains $\mathcal{D}_a$, $\mathcal{D}_b$ with $L_a \sim 665$ km and $L_b \sim 882$ km were used in the study of regional transport of atmospheric pollutants (Yamada et. al., 1989; Fast and Zhong, 1998) and $\mathcal{D}_c$ with $L_c \sim 1665$ km is used in the operational meteorological analysis of México (SMN, 2002). The solution eq. (3) together with the equation of state and the boundary condition $p = p_0$ at $x = y = z = 0$ is

$$p_s(r) = p_0 e^{-ba(1-a/r)} \qquad\qquad b \equiv g/\mathcal{R}T_0$$

with $r = [x^2+y^2+(z+a)^2]^{1/2}$, which is the pressure field on the *whole* terrestrial sphere. This shows that the eq. (3) in the xyz system is valid on *any* domain $\mathcal{D}(L)$ indeed. Hence we get the pressure value on the isobar f that passes through the $z-$axis point $(\xi = 0, z = z_0)$, namely,

$$p(x = y = 0, z_0) = p_0 \exp[-bz_0/(1 + z_0/a)]. \qquad (4)$$

The solution of eq. (2) with $p^0 = p_0$ at $x = y = z = 0$ is

$$p^0(z) = p_0 e^{-bz}.$$

Consider the relative error of p^0 on the terrestrial sphere, $r = a$, as a function of the distance ξ between the corner of $\mathcal{D}(L)$ and the origin $x = y = z = 0$, $\xi = \sqrt{2}L$, namely, $\Delta p^0(\xi) = (p^0/p - 1)100$ with

$$z = -a + \sqrt{(a + z_0)^2 - \xi^2}$$

and p is given by (4). The values $T_0 = 300$ ^{o}K, $\mathcal{R} = 287$ J/kg^{o}K, $p_0 = 1013$ mb, g =9.8 ms^{-2} and $a = 6378$ km yield $b = 0.11382$ km^{-1}. The error Δp^0 is less than 20% for $\xi \leq \xi^0_{\rm max} \sim 160$ km and increases rapidly from 20 to 300% as ξ goes from $\xi^0_{\rm max}$ to 400 km. This suggests that $\xi^0_{\rm max} \lesssim 160$ km provides the upper bound $L^0_{\rm max} \lesssim 113$ km for the reliability domain $\mathcal{D}(L^0_{\rm max})$ of the approximate equation (2).

3. Equations from map projections

To analyze the role of map projections consider the formal definition of the projection coordinates $x_p y_p H_p$. Let $x_p y_p$ be a cartesian coordinate system on a projection plane $\mathcal{P}$ which is normal to the H_p axis. The conformal projection of a point (λ, ϕ) on the terrestrial sphere is the point (x_p, y_p) given by a pair of projections equations

$$x_p = P_x(\lambda, \phi) \qquad\qquad y_p = P_y(\lambda, \phi). \qquad (5)$$

Usually, the center (λ_c, ϕ_c) of the domain $\mathcal{D}$ is projected on the origin of the $x_p y_p$ system, $x_p(\lambda_c, \phi_c) = y_p(\lambda_c, \phi_c) = 0$, and the eastward parallel circle and the northward meridian on (λ_c, ϕ_c) are projected on the positive x_p and y_p axes, respectively. If a point in physical space has spherical coordinates (λ, ϕ, r) its coordinates x_p, y_p are given by (5) and H_p is defined by $H_p = r - a$. Thus we have four equivalent sets of coordinates to define the position of a parcel, namely, (x, y, z), (X, Y, Z) and (λ, ϕ, r) which have a simple geometrical interpretation in physical space while (x_p, y_p, H_p) are coordinates in an abstract space. The governing equations in coordinates $x_p y_p H_p$ are obtained from the equations in spherical coordinates (Haltiner, 1971). The solution of these equations with the pertinent boundary and initial conditions generates the meteorological fields in the $x_p y_p H_p$ space but in order to analyze such fields in physical space we have to apply the pertinent coordinate transformations to obtain the fields in coordinates x, y, z or λ, ϕ, r, for details see Nuñez (2002). However, the approximation

$$x_p \sim x \quad y_p \sim y \quad H_p \sim z$$

is used to work on the cartesian coordinate system xy with the expectation that the map projection considers the spherical shape of the earth (see also Perkey, 1986). If this approximation is used we to solve equations like those reported by Haltiner (1971) where xyz play the role of $x_p y_p H_p$. For instance, the stereographic projection yields horizontal momentum equation

$$\frac{du_p^*}{dt} - v_p^* \left(f + \frac{y u_p^* - x v_p^*}{2a^2} \right) = \frac{w_p^*}{a} \left[(1 + \sin\phi)\, \Omega y - u_p^* \right] - m\alpha \frac{\partial p_p^*}{\partial x} \qquad (6)$$

(Haltiner, 1971). We see that the horizontal momentum equations have no gravity-force term and therefore such equations are similar to the equation (2), a conclusion verified by the solution for an isothermic and hydrostatic atmosphere. In this case the pressure field with $p_p^* = p_0$ (1013 mb) at $x = y = z = 0$ is $p_p^* = p_0 e^{-baz/(z+a)}$ which is essentially the pressure field p^0 from the equations (2) if we consider $|z| \ll a$. Thus we can say that the map-projection equations like (6) are valid on $\mathcal{D}^0 \lesssim 200{\times}200$ km^2.

4. Summary

In agreement with McVittie (1948) and Dutton (1976), the results of section 2 suggest that the eq. (2) is valid on a horizontal domain $\mathcal{D}(L^0_{\max}) \lesssim 200{\times}200$ km^2. Of course, the example of an hydrostatic atmosphere ignores important factors controlling a real flow such as the stratification and, mainly, the time evolution which can generate important qualitative differences between the flows from eqs. (2) and (3) because of their nonlinearity. However, the numerical modeling of some mesoprocesses requires the use of a large domain $\mathcal{D}(L)$ (i) to include the influence of propagating synoptic disturbances on the regional weather and (ii) to reduce the error from the lateral boundary conditions inherent to limited-area modeling. In principle, this conflict can be solved with the use of the momentum equation (3), which is valid on any domain $\mathcal{D}(L)$. In practice, $\mathcal{D}(L)$ will be limited by (i) the available data to define the initial and boundary conditions and (ii) the computational resources. For example, if $L = 500$ km and the height of the troposphere on the terrestrial sphere is $H = 18$ km we have to use a tridimensional model region with a height $H_M = |z|_{\max} + H \sim 57.3$ km, where

$$|z|_{\max} = \left| -a + \sqrt{a^2 - 2L^2} \right|$$

and $a = 6378$ km, which increases significantly the computational cost and probably the data from global prediction models are insufficient to define initial conditions.

If xyz $(\widehat{\mathbf{x}}\widehat{\mathbf{y}}\widehat{\mathbf{z}})$ are denoted by $x^1 x^2 x^3$ $(\widehat{\mathbf{x}}_1 \widehat{\mathbf{x}}_2 \widehat{\mathbf{x}}_3)$, respectively, and we set $\tilde{x}^1 = x^1$, $\tilde{x}^2 = x^2$, $\tilde{x}^3 = \tilde{x}^3(x^1, x^2, x^3, t)$, the contravariant form of the exact $\mathbf{g}$ is

$$\mathbf{g} = g^j \hat{\mathbf{x}}_j = g^j \frac{\partial \tilde{x}^i}{\partial x^j} \tau_i$$

where $g^1 = -ga^2 x r^{-3}$, $g^2 = -ga^2 y r^{-3}$, $g^3 = -ga^2 (z + a) r^{-3}$ and τ_i are the covariant vectors from the $\tilde{x}^j$'s. Hence, the contravariant form of eq. (3) is

$$\frac{\partial \tilde{u}^i}{\partial t} = -\tilde{u}^j \tilde{u}^i_{,j} - \tilde{G}^{ij} \theta \frac{\partial \pi}{\partial \tilde{x}^j} + g^j \frac{\partial \tilde{x}^i}{\partial x^j} - 2\varepsilon^{ijl} \Omega_j \tilde{u}_l, \quad (7)$$

where frictional forces are neglected, while the contravariant form of eq. (2) is

$$\frac{\partial \tilde{u}^i}{\partial t} = -\tilde{u}^j \tilde{u}^i_{,j} - \tilde{G}^{ij}\theta \frac{\partial \pi}{\partial \tilde{x}^j} - \frac{\partial \tilde{x}^i}{\partial x^3} g - 2\varepsilon^{ijl}\Omega_j \tilde{u}_l, \quad (8)$$

(Pielke, 1984). The practical limitations discussed above impose the use of a domain $\mathcal{D}(L)$ with $L \leq 500$ km. In this case we can use the linear approximation $g^1 \sim -gx/a$, $g^2 \sim -gy/a$, $g^3 \sim -g$ in eq. (7).

The horizontal momentum equations reported in some references have terms with g but it does not come from the use of the correct gravity acceleration $\mathbf{g}$ (2). For instance, from the equation (8), $\sigma_z = s(z - z_G)/(s - z_G)$, the hydrostatic relation and the chain rule Pielke (1984) obtains

$$\begin{aligned} \frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^j \frac{\partial \tilde{u}^1}{\partial \tilde{x}^j} + \overline{\tilde{u}^j \frac{\partial \tilde{u}^1}{\partial \tilde{x}^j}} \quad &= \quad -\theta \frac{\partial \pi}{\partial \tilde{x}^1} + g \frac{\sigma - s}{s} \frac{\partial z_G}{\partial x} \\ &\quad -\hat{f}u^3 + fu^2 \end{aligned}$$

where the terms with g^1, g^2 or their linear approximation are absent.

5. References

Dudhia J., Gill D., Guo Y.-R., Hansen D., Manning K. and Wang W., 1999, *PSU/NCAR Mesoscale Modeling System Tutorial Class Notes and Users' Guide (MM5 Modeling System Version 2 with an introduction to version 3)*. http:// www.mmm.ucar.edu/ mm5/ mm5-home.html.

Dutton J. A., 1976, *The Ceaseless Wind: An introduction to the theory of atmospheric motion*, (McGraw-Hill, New York).

Fast J. D. and Zhong S., 1998, *J. Geophys. Res.* **103,** pp. 18927-18946.

Haltiner G. J., 1971, *Numerical Weather Prediction* (Wiley and Sons, New York).

McVittie G. C., 1948, Quat. J. mech. Appl. Math. **1**, 174-195.

Nuñez M. A., 2002, Nuovo Cimento C **25**, 13-34; *Domain of Validity of some Atmospheric Mesoscale Models*, Internal Report, Physics Dept., Universidad Autonoma Metropolitana Iztapalapa,.

Perkey D. J., 1986, *Formulation of Mesoscale Numerical Models*, in *Mesoscale Meteorology and Forecasting*, edited by P. S. Ray (American Meteorological Society, Boston).

Pielke R. A., 1984, *Mesoscale Meteorological Modeling* (Academic Press, New York).

Pielke R. A., 1994, *The Status of Mesoscale Meteorological Models*. In *Planning and Managing Regional Air Quality*, edited by P. A. Solomon and T. A. Silver (Pacific Gas & Electric Co. and Lewis Publishers, San Ramon, CA).

Pielke R. A. and Coauthors, 1992, *Meteorol. Atmos. Phys.* **49,** 69-91. *RAMS The Regional Atmospheric Modeling System Technical Description DRAFT*, http:// www.atmet.com/html/ docs/ rams_techman.pdf.

Servicio Meteorológico Nacional, *Pronóstico Numerico Regional (Modelo MM5)*, http:// smn.cna.gob.mx/ productos/mm5/ htm/pag7773.htm

Yamada T., *J. Met. Soc. Japan*, **59** (1981) 108-122.

Yamada T. and Bunker S., 1988, J. Appl. Meteor. **27,** 562-578.

Yamada T., Kao J. C.-Y. and Bunker S., 1989, *Atmos. Environ.* **23,** 539-554.

Xue M., Droegemeler K. K., Wong V., Shapiro A. and Brewster K., 1995, *Advanced Regional Prediction System (ARPS) Version 4.0 User's Guide*. http:// wwwcaps.ou.edu/ARPS/.