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MOIST NWP EQUATIONS AND MISSING TERMS

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1. INTRODUCTION

Most large-scale NWP and climate models are based upon the hydrostatic primitive equations and use a pressure-based terrain-following coordinate in the vertical. Phillips (1957) introduced the normalised pressure coordinate $\sigma = p / p_s$ where p is the pressure and the subscript s denotes the surface value, hence terrain following. The sloping surfaces are not ideal away from the surface so it is usual to allow the coordinate surfaces to gradually flatten until they become constant pressure surfaces, a hybrid system, Simmons and Burridge (1981). Kasahara (1974) derived the dry hydrostatic primitive equations for a generalized vertical coordinate and his methodology can be found in standard textbooks for NWP. These equations are usually modified to include moist effects by including transport equations for moist quantities and using virtual temperature rather than temperature in the pressure gradient terms and the adiabatic conversion terms, if present. It is not usually acknowledged that for the moist equations the continuity equation also changes as will be explained in the next section. These changes take into account pressure changes due to moist effects, which are not usually allowed for in large-scale models. The model used by Méteo-France does include an option to include pressure changes due to moist fluxes. This is described, in an unpublished manuscript written before 1990 . The following analysis shows how these moist terms arise naturally if the equations are derived from first principles.

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The present work arose out of investigations into the use of moist variables in a new model at the Met Office. It was not always clear which model components (and data used by the model) were meant to be mixing ratios or specific quantities. Clarifying these issues revealed what may be an oversight in the usual derivation of the moist hydrostatic primitive equations.

2. THE CONTINUITY EQUATION

The continuity equation for dry air with density ρ_{dry} is:

$$\frac{\partial \rho_{dry}}{\partial t} + \nabla \bullet \left(\rho_{dry} \underline{v} \right) = 0 \qquad \text{or}$$

$$\frac{D\rho_{dry}}{Dt} + \rho_{dry} \, \nabla \bullet \underline{v} = 0 \, .$$

Following Kasahara (1974), transforming to a system in a generalized vertical coordinate η, the divergence term becomes

$$\nabla \bullet \underline{v} = \nabla_z \bullet \underline{v} + \frac{\partial w}{\partial z} = \nabla_\eta \bullet \underline{v} + \frac{D}{Dt}\left(\ln \left| \frac{\partial z}{\partial \eta} \right| \right) + \frac{\partial \dot{\eta}}{\partial \eta}$$

and the continuity equation then becomes

$$\frac{D}{Dt}\left(\ln \left| \rho_{dry} \frac{\partial z}{\partial \eta} \right| \right) + \nabla_\eta \bullet \underline{v} + \frac{\partial \dot{\eta}}{\partial \eta} = 0 \, .$$

In deriving the hydrostatic primitive equations, we normally use the hydrostatic equation in the form $\rho_{dry} \frac{\partial z}{\partial \eta} = -\frac{1}{g} \frac{\partial p}{\partial \eta}$ and the first term in the above equation becomes $\frac{D}{Dt}\left(\ln \left| \frac{\partial p}{\partial \eta} \right| \right)$ giving

$$\frac{D}{Dt}\left(\ln \left| \frac{\partial p}{\partial \eta} \right| \right) + \nabla_\eta \bullet \underline{v} + \frac{\partial \dot{\eta}}{\partial \eta} = 0 \, . \qquad (1)$$

However, the hydrostatic equation for a moist atmosphere is

$$\rho\frac{\partial z}{\partial \eta} = -\frac{1}{g}\frac{\partial p}{\partial \eta}$$

where

$$\rho = \rho_{dry} + \rho_v + \rho_{cl} + \rho_{cf} = \rho_{dry} + \sum_X \rho_X \, .$$

The subscripts v, cl, cf denote water vapour, cloud water and cloud ice respectively and the summation is over the moist quantities. We could add any number of constituents without affecting the subsequent analysis. Each of the moisture quantities has a transport equation of the form

$$\frac{Dm_X}{Dt} = S_X \quad ; X = v, cl, cf$$ where

$m_X = \dfrac{\rho_X}{\rho_{dry}}$ are the mixing ratios and S_X are the source/sink terms. Specific quantities may be used instead of mixing ratios. Returning to our hydrostatic equation, we replace ρ by

$$\rho_{dry} + \sum_X \rho_X = \rho_{dry}\left(1 + \sum_X m_X\right)$$

which leads to

$$\rho\frac{\partial z}{\partial \eta} = \rho_{dry}\left(1 + \sum_X m_X\right)\frac{\partial z}{\partial \eta} = -\frac{1}{g}\frac{\partial p}{\partial \eta}$$

and our revised first term of the continuity equation becomes

$$\frac{D}{Dt}\left[\ln\left|\frac{1}{\left(1+\sum_X m_X\right)}\frac{\partial p}{\partial \eta}\right|\right] =$$

$$\frac{D}{Dt}\left(\ln\left|\frac{\partial p}{\partial \eta}\right|\right) - \frac{1}{\left(1+\sum_X m_X\right)}\sum_X \frac{Dm_X}{Dt} \, .$$

Using the transport equation for m_X and $1/\left(1+\sum_X m_X\right) = \rho_{dry}/\rho$, our revised continuity equation is

$$\frac{D}{Dt}\left(\ln\left|\frac{\partial p}{\partial \eta}\right|\right) + \nabla_\eta \bullet \underline{v} + \frac{\partial \dot{\eta}}{\partial \eta} = \frac{\rho_{dry}}{\rho}\sum_X S_X \, . \quad (2)$$

Thus comparing equations (1) and (2) we see that the moist hydrostatic primitive equations should include the effects of moist processes as source/sink terms in the continuity equation. These terms are omitted from most moist hydrostatic primitive equation models and also in non-hydrostatic models having the same derivation of the continuity equation. The effect of these terms is likely to be small in relation to the pressure tendencies and are probably safely ignored for large-scale weather forecasting. For smaller scales (i.e. models at very high resolution) the local pressure tendency might be dominated by the precipitation flux so these terms become more important.

In the tropics, where there is almost continuous evaporation at the sea-surface, the terms act in the same sense all the time and the evaporation will peak around local midday. These terms may therefore be more important for long climate runs.

Including these terms in a semi-implicit model should be relatively straightforward since the source/sink terms should be available from the various physical parametrizations. In an explicit scheme using fractional timestepping, the source/sink terms would need to be apportioned over the timestep for each calculation of the pressure tendency and vertical velocity.

3. REFERENCES

Kasahara, A. (1974). Various vertical coordinate systems used for numerical weather prediction. *Mon. Wea. Rev.*, **102**, 509-522.

Phillips, N. A. (1957). A coordinate system having some special advantages for numerical forecasting. *J. Meteor.*, **14,** 184-185.

Simmons, A.J. and Burridge, D.M. (1981). An energy and angular momentum conserving vertical finite difference scheme and hybrid vertical coordinates. *Mon. Wea. Rev.*, **109**, 758-766.