4.2 THE CONCEPT OF POTENTIAL MOMENTUM AND ITS APPLICATION FOR THE EXTRATROPICAL CIRCULATION

Pablo Zurita-Gotor*
SUNY Stony Brook, Stony Brook, New York

R.S. Lindzen
MIT, Cambridge, Massachusetts

1. MOTIVATION

The development of Lagrangian-mean theories has greatly enhanced our understanding of the ways in which eddies and mean flows interact. In the qg limit, the Transformed Eulerian Mean (TEM) formalism provides a useful approximation to the Lagrangian mean balance (Edmon et al, 1980).

In this framework, the EP vectors represent the net eddy flux of (easterly) momentum, while the eddy PV flux appears as an explicit momentum forcing:

$$\frac{\partial \overline{U}}{\partial t} - f_0 \overline{v}^* = \overline{v'q'} - \alpha_M \overline{U} \tag{1}$$

This is akin to the zonal momentum equation for 2D flow, except that in that case the forcing by the residual circulation disappears. This has important conceptual implications. Consider the momentum balance for the equilibrated (i.e, dropping the time derivative) 2D problem. The eddy PV flux is the only dynamical forcing of momentum in that case, and must be balanced in equilibrium by friction. On the other hand, the eddy PV flux can also be linked to the *local* rate of eddy enstrophy dissipation:

$$\alpha_M \overline{U} = \overline{v'q'} \approx -\alpha_M \frac{\overline{q'^2}}{\overline{q}_y} \tag{2}$$

Hence, for the *equilibrated*[1] flow, a local balance exists between the zonal momentum forcing and the eddy enstrophy dissipation. A straightforward implication of this balance is that, for downgradient eddy PV fluxes, the equilibrium PV gradient must be negative (positive) over regions of westerly (easterly) mean flow acceleration.

However, things are more complicated in the 3D problem, in which there is an additional forcing of momentum by a remotely forced[2] residual circulation. Zurita-Gotor and Lindzen (2003) get rid of this term by integrating vertically, and thus arrive to a local balance for the vertically integrated momentum. A direct implication of this balance is that the eddies cannot eliminate the surface temperature gradient over latitudes with surface westerlies.

This suggests that the vertical redistribution of momentum by the residual circulation makes the 3D momentum balance essentially non-local. However, we show in this paper that it is possible to recapture in the 3D case a local balance similar to equation 2. To this end, we derive a new formalism in which the thermal structure is represented as 'potential momentum'. Then, a local balance analogous to the 2D one exists, provided that we recognize that the eddy PV flux forces both the physical and potential momentum. As this formalism makes explicit the connection between temperature and momentum, it could also provide a useful framework in which to study a number of unsolved geophysical problems.

2. POTENTIAL MOMENTUM

We start with the TEM qg thermodynamic equation:

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{w}^* \Theta_z = -\alpha_T (\overline{\theta} - \overline{\theta}_R) \tag{3}$$

where Θ_z is the basic state stratification and θ_R an equilibrium profile toward which the flow is relaxed.

Consider a latitude $y_0 = 0$, such that $\overline{v}^* = 0$. We multiply equation 3 by $-f_0/\Theta_z$, integrate meridionally between y_0 and y and differentiate with respect to z. Applying continuity of the residual circulation, equation 3 can then be written:

$$\frac{\partial \overline{M}}{\partial t} + f_0 \overline{v}^* = -\alpha_T (\overline{M} - \overline{M}_R) \tag{4}$$

where we define the *potential momentum* M as:

$$M = -\int_0^y \frac{\partial}{\partial z} \left(\frac{f_0}{\Theta_z} \theta \right) dy' \tag{5}$$

Note that M has momentum units and that equation 4 formally looks like a momentum equation. Comparing equations 1 and 4 we can see that the role

* Corresponding author address: Pablo Zurita-Gotor, ITPA/MSRC, SUNY Stony Brook, Stony Brook, NY 11794-5000; email: pzurita@alum.mit.edu.

[1]But not for the transient problem.

[2]Note that the Eliassen-Kuo equation for $\overline{v}^*$ is elliptic

of the mean circulation $\overline{v}^*$ is to transform potential into physical momentum. Adding both equations together this term disappears and we get:

$$\frac{\partial}{\partial t}\left(\overline{M}+\overline{U}\right)+\alpha_T\left(\overline{M}-\overline{M}_R\right)+\alpha_M\overline{U}=\overline{v'q'} \quad (6)$$

We thus recover a local balance, analogous to the 2D case, in which $\overline{v'q'}$ is the only dynamical forcing of momentum. The primary difference is that now the eddy PV flux forces the total momentum $\overline{U}+\overline{M}$, and not just $\overline{U}$. Hence, the *total* momentum can be thought to be determined through a local balance, while the remotely forced circulation $\overline{v}^*$ only determines the partition between $\overline{M}$ and $\overline{U}$, which must be such that thermal wind is satisfied:

$$\frac{\partial^2\overline{M}}{\partial y^2}=\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\overline{U}}{\partial z}\right) \quad (7)$$

Equation 6 can also be linked to the conservation of pseudomomentum $\overline{A}\approx 1/2\ \overline{q'^2}/\overline{q_y}$:

$$\frac{\partial}{\partial t}\left(\overline{M}+\overline{U}\right)=\overline{v'q'}-D_{mean} \quad (8)$$

$$\frac{\partial}{\partial t}\overline{A}=-\overline{v'q'}-D_{eddy} \quad (9)$$

where for our simple form of forcing:

$$D_{mean}=\alpha_M\overline{U}+\alpha_T(\overline{M}-\overline{M}_R) \quad (10)$$

$$D_{eddy}\approx\frac{1}{\overline{q}_y}\left(\alpha_M\overline{\xi'q'}-\alpha_T\overline{m'_yq'}\right) \quad (11)$$

It is in this form that the role of the PV flux as an eddy drag appears most clearly, as it generates eddy pseudomomentum at the expense of the deceleration of the mean flow. The equilibrium is then determined through the local[3] balance $D_{mean}+D_{eddy}=0$ between the forcing of zonal momentum and eddy enstrophy dissipation, much like in the 2D case.

Note that equation 6 is simply the y-integral of the qgpv equation. For symmetric 2D flow, momentum provides a full description and vorticity is redundant. Likewise, symmetric 3D flow can be encapsulated in terms of momentum alone, provided that we define a momentum expression of the baroclinic term. This is essentially what the potential momentum represents.

Physical interpretation

From its definition, it is obvious that the potential momentum is simply the zonal momentum profile that would produce the same PV distribution as the stretching term in the full 3D basic state[4]:

$$\frac{\partial\overline{M}}{\partial y}=-\frac{\partial}{\partial z}\left(\frac{f_0}{\Theta_z}\overline{\theta}\right) \quad (12)$$

If the stratification were constant and equal to its reference value Θ_z everywhere (i.e., $\overline{\theta}_z(y,z)=0$), the potential momentum would be zero. $\overline{M}$ thus essentially represents deviations of the isentropic thickness from its reference value. Specifically, we can see from equation 12 that an easterly (westerly) shear in $\overline{M}$ reflects isentropic thicknesses smaller (larger) than this reference value.

By construction, $\overline{M}(y_0)=0$. Moreover, if the reference stratification $\Theta_z(z)$ is properly defined, $\overline{\theta}_z$ has no mean and equation 5 should integrate to zero, implying that $\overline{M}(y_L)=0$ as well. Taking this into account, it is easy to see (refer to figure 1) that easterly (westerly) values of $\overline{M}$ are associated with isentropes that open up (close down) with latitude. The former is the case in the extratropical troposphere because the isentropic slope increases with height. This potential-momentum easterly jet gives a negative contribution to the interior PV gradient.

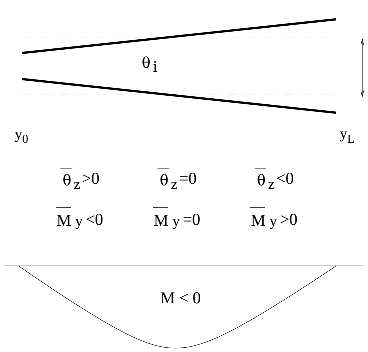

Figure 1: *Sketch illustrating the $\overline{M}$ distribution resulting from isentropes that open up with latitude.*

It is useful to regard the lower boundary as an isothermal surface (Bretherton, 1966), which requires the inclusion of a delta-function interior PV gradient right above it. This delta-function PV gradient results from the $\overline{M}$ contribution to the total

[3] But note that $\overline{M}$, as defined in 5, is a non-local function of the thermal structure; in a sense, the non-local nature of this problem has been hidden in the definition of $\overline{M}$.

[4] The wording *potential* was chosen to emphasize the fact that this potential momentum can be converted into physical momentum through an adiabatic redistribution of mass (i.e., a mean circulation $\overline{v}^*$). However, potential momentum is not conserved, and should not be confused with potential vorticity. We could have alternatively defined potential momentum as the zonal momentum that produces the same *full* PV distribution (i.e, $\overline{U}+\overline{M}-\int(f_0+\beta y)dy$), which is actually conserved in an adiabatic redistribution of mass.

PV gradient, and can be easily derived by generalization of equation 5:

$$\overline{M}(y,z) = -\frac{f_0}{\Theta_z}\int_0^y \left(\overline{\theta} - \Theta_0\right)\delta(z)dy' \quad at\ z=0 \tag{13}$$

where Θ_0 is the uniform temperature of the isothermal surface. Defining this temperature as the mean surface temperature between y_0 and y_L, we can also make $\overline{M}$ go to zero at both surface endpoints.

When the surface temperature decreases with latitude, equation 13 also produces an easterly potential momentum and a negative delta-function contribution to $\overline{q}_y$. Physically, this can also be interpreted in terms of the isentropes opening up with latitude if we regard the disappearance of the isentrope θ_i equatorward of the latitude y_i as a jump in thickness between 0 and its interior value at that latitude.

Finally, we point out that in the previous discussion and in figure 1 it was implicitly assumed that the isentropic layers were quasi-horizontal. This makes the interpretation easier because we can identify integrating at constant height with moving along an isentrope. However, this assumption is not strictly valid for planetary dynamics because the isentropic slope is not small: in reality, the integral in equation 5 is taken along different isentropic layers. While this complicates the interpretation, the basic concepts should still be valid.

3. GENERAL CIRCULATION

We next discuss the different terms contributing to the generation and destruction of $\overline{M}$. First, it should be noted that when the surface delta function is included, there is a constraint on the vertically integrated potential momentum. From equations 5, 13:

$$\int_0^\infty \overline{M}dz = -\frac{f_0}{\Theta_z}\int_0^y \left(\overline{\theta}_\infty - \Theta_0\right)dy = \int_0^\infty \overline{M}_R dz$$

In a seminfinite domain $\overline{\theta}_\infty$ would be the temperature at infinity, which cannot be changed by the dynamics (assuming a finite mixing depth). Alternatively, in the presence of an upper rigid lid, $\overline{\theta}_\infty$ would be the constant temperature of the top boundary when the boundary temperature gradient is included as an interior delta-function PV gradient. In both cases, this equation implies that the vertically integrated potential momentum is constant.

Physically, the reason for this constraint is that the sum of the isentropic thicknesses of all layers must add up to a constant because we assumed that potential temperature was fixed at two reference heights. Hence, the eddies and diabatic processes can only redistribute the mass between the different isentropic layers. For instance, as the surface temperature gradient is reduced an isentropic layer that is initially absent at a given latitude may acquire a finite thickness at the expense of the interior layers.

According to equation 4, potential momentum can be created or destroyed diabatically, or it can be transformed into physical momentum through the residual circulation. However, because each of the terms in this equation integrates independently to zero in the vertical (including delta surface contributions), the generation and conversion terms only redistribute potential momentum vertically, but conserve the net potential momentum of the column.

Diabatic processes transfer mass between the different isentropic layers, thus changing the distribution of isentropic thicknesses at a given latitude. This results in a vertical redistribution of potential momentum, with no net creation or destruction along the column. In particular, heating at low latitudes and cooling at high latitudes is equivalent to an upward transfer of potential momentum for a stably stratified fluid. This is illustrated in figure 2, which shows two adjacent isentropic layers θ_1 and θ_2, with $\theta_2 > \theta_1$. Low latitude heating results in a transfer of mass from θ_1 to θ_2, while the reverse is true for high latitude cooling. This tends to generate easterly potential momentum along the lower isentrope, and compensating westerly potential momentum aloft.

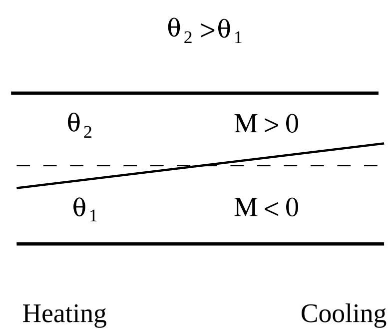

Figure 2: *Sketch illustrating the redistribution of* $\overline{M}$ *resulting from differential heating.*

Additionally, there is a conversion between potential and physical momentum through the residual circulation. In particular, a poleward circulation $\overline{v}^* > 0$ converts potential momentum into physical momentum. As mass is moved poleward along the layer, the poleward thickness increases and there is a generation of easterly potential momentum (fig. 1). If the mass rearrangement is adiabatic, the total momentum $\overline{M} + \overline{U}$ must be conserved, which requires an equal generation of westerly physical momentum via the Coriolis acceleration. Hence, in this framework the westerly acceleration characteristic of the equilibration of baroclinic instability comes up naturally as the reduction of the surface temperature gradient (i.e., the depletion of the easterly potential momentum at the surface) must be compensated by an interior generation of easterly $\overline{M}$ and westerly $\overline{U}$.

The circulation can then be described as follows. Diabatic processes force easterly potential momentum at the surface, which gives a negative $\overline{q}_y$ and renders the basic state unstable. As the waves grow, the downgradient PV fluxes produce easterly drag in the interior, and westerly drag at the surface[5]. However, the massless surface undergoes no net acceleration, as this eddy drag is exactly balanced by the return flow of the residual circulation $f_0\overline{v}^*$. Nevertheless, the circulation does deplete the surface potential momentum, and reduces the surface temperature gradient. At the same time, the interior branch of this circulation produces compensating easterly $\overline{M}$ in the interior, as well as a westerly acceleration in $\overline{U}$. In equilibrium, the former is balanced diabatically, and the latter is balanced by easterly eddy drag $\overline{v'q'}$ and mechanical friction. We illustrate this in more detail in poster P2.4, where we apply potential momentum diagnostics to the equilibration of Charney waves.

4. CONCLUSION

In this paper we have introduced the concept of potential momentum and proposed a new, momentum-based framework in which to understand the extratropical circulation. To conclude, we discuss some of the possible advantages of this formulation:

1) In our view, it provides a simpler and more elegant description of the zonally-averaged quasi-geostrophic dynamics. Though space prevents us from a more thorough discussion, many classical results, like the non-interaction theorem, are most naturally expressed in this framework.

2) It also provides a simpler framework in which to understand wave-mean flow interaction. For instance, equations 8-11 naturally relate the mean flow imbalance to eddy amplitude (i.e., the mean flow can only be pushed off-balance in the presence of a finite amplitude wave). This could have applicability for closure theories and eddy parameterization. Moreover, the local nature of the wave-mean flow interaction supports the construction of diffusive theories.

3) The potential-momentum formalism provides a natural generalization of the 2D dynamics. For instance, we argued that for forced-dissipative 2D flow, the mean-flow acceleration must be westerly (easterly) over regions of negative (positive) PV gradient. Similarly, in the 3D problem there is an easterly (westerly) *potential momentum* acceleration in the interior (at the surface).

4) In this framework momentum and 'temperature' can be directly compared, thus offering a more coherent view than a traditional description in terms of eddy heat and momentum fluxes. In particular, the surface temperature gradient appears as a momentum source, whose depletion must be accompanied by a net westerly acceleration. The role played by friction and the barotropic governor in the baroclinic equilibration are also most transparent in this framework (see figure 3). This will be discussed in more detail in a different presentation (poster P2.4).

5) For the same reasons, the potential momentum formalism is appealing for problems involving the combined effects of thermal and mechanical forcing. Examples for which this could be useful are tropical-extratropical interactions or the ACC dynamics.

6) The concept of potential momentum emphasizes latitudinal variations in isentropic thickness, an aspect of the thermal structure that has received relatively little attention in the literature. This suggests that additional forms of thermal forcing other than the pure Equator-to-Pole differential heating could be important. In other words, processes locally affecting the stratification could also have a global impact on climate. The derived formalism provides a natural framework in which to understand the role of subtropical moist convective adjustment and/or polar inversions for global climate.

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[5]By this, we mean the masless PV sheet of qg theory.

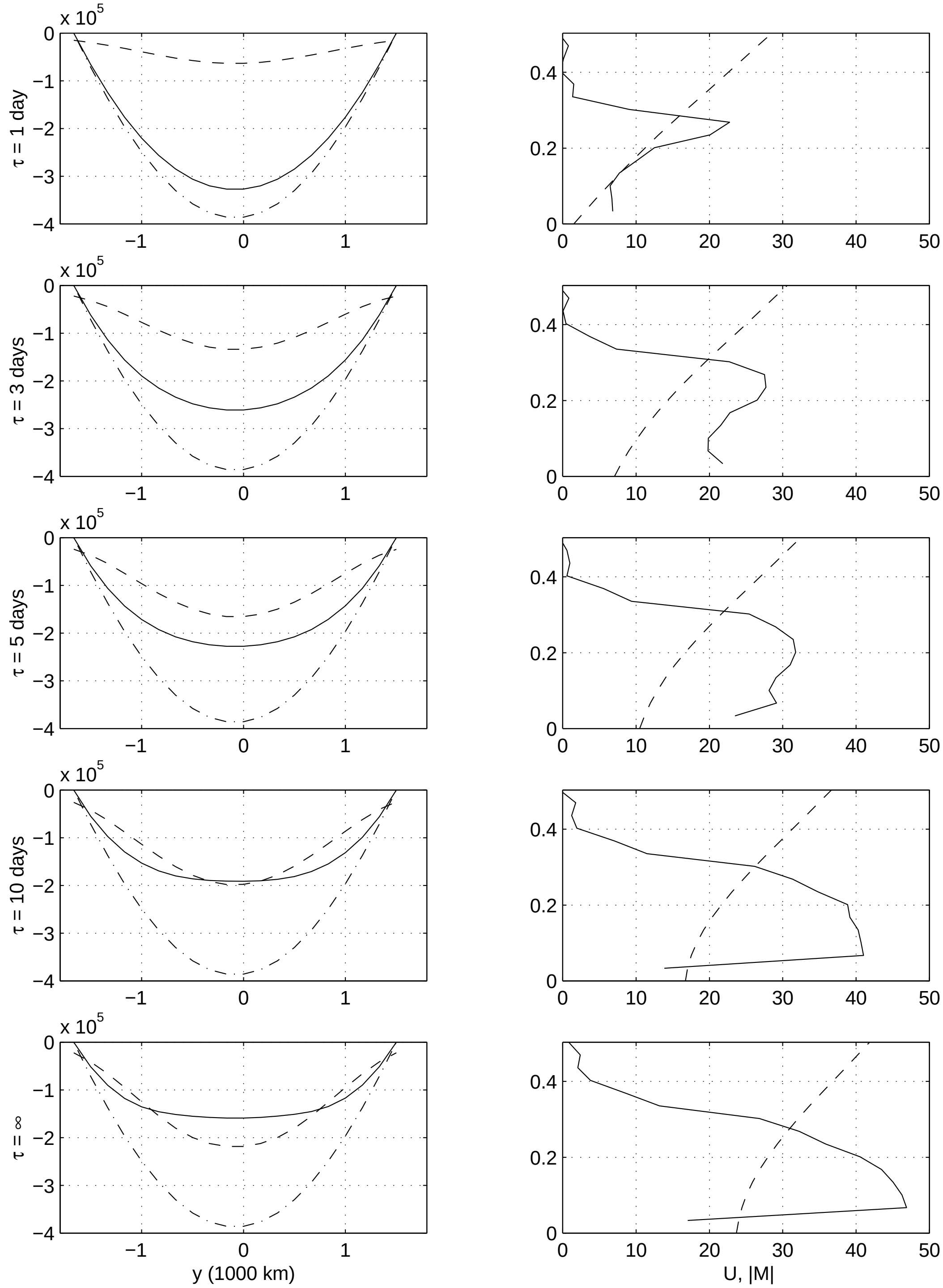

Figure 3: *This figure illustrates the role of friction on the baroclinic equilibration using potential momentum diagnostics for the idealized model described in P2.4. The right panels show the (absolute value) physical (dashed) and potential (solid) momentum jets, and the left panels the partition between the surface (solid) and interior (dashed) potential momentum; each row corresponds to a different value of friction. As friction is reduced, both the* $\overline{U}$ *and* $\overline{M}$ *jets are enhanced. The generation of interior* $\overline{M}$ *occurs at the expense of depleting the surface* $\overline{M}$ *reservoir, or reducing the surface temperature gradient. See poster P2.4 for details.*