2.2 INERTIA-GRAVITY-WAVE GENERATION BY BALANCED MOTION

J. Vanneste*

University of Edinburgh, UK

1. INTRODUCTION

The rapid rotation and strong stratification of the atmosphere and oceans at mid-latitudes lead to a clear separation between the advective time scales and the inertia-gravity-wave (IGW) time scales. This time-scale separation, characterized by a small parameter $\epsilon \ll 1$ (essentially the Rossby number), implies that the flow can remain close to a balanced state, free of IGWs. Such a state is best thought of as a manifold of reduced dimensionality in the state space of the system — a slow manifold. Balanced models (e.g. the quasi-geostrophic model) then result from the projection of the primitive equations onto such a manifold, and initialization procedures amount to the projection of initial data.

Using power-series expansions in ϵ, one can in principle obtain a hierarchy of slow manifolds, $\mathcal{M}_n$ say, by truncation at some power ϵ^n. Trajectories of the primitive equations then remain (for a finite time) within an $O(\epsilon^n)$ distance of $\mathcal{M}_n$. Were this procedure to converge, one could define *the* slow manifold $\mathcal{M}_\infty$, an exactly invariant manifold on which the motion is entirely devoid of IGWs.

However, it has become clear that such an exactly invariant slow manifold does not exist in general, and that balanced motion, however well initialized, spontaneously generates IGWs. This is consistent with of the possible definition of the (approximately invariant) $\mathcal{M}_n$ for arbitrary n because the expansion procedures are divergent, and because the amplitude of the IGWs that are generated is smaller than any order ϵ^n. Typically, one expects wave amplitudes to be exponentially small, scaling like $\exp(-\alpha/\epsilon)$ for some $\alpha > 0$.

The conclusions just outlined have been drawn using a combination of numerical and analytical results, mostly for low-order models (e.g. Bokhove & Shepherd 1996, Camassa & Tin 1996 and references therein). The analytic results are mainly upper bounds on IGW amplitudes. What appears to be lacking, however, are explicit estimates of these amplitude in the regime most relevant to geophysical fluids, namely the quasi-geostrophic regime, with small Rossby and Froude numbers, both of a similar order of magnitude. This abstract reports on the asymptotic derivation of such estimates in two simple models.

To capture the amplitude of IGWs spontaneously generated by balanced motion, the techniques of exponential asymptotics, or asymptotics beyond all orders, must be used. By associating the generation of IGW to a Stokes phenomenon, these reveal the importance of considering complex values of the time variable t and, specifically, of identifying the complex values of t for which the balanced motion becomes singular.

2. LORENZ'S 5-COMPONENT MODEL

The system of ordinary differential equations

$$\dot{u} = -vw + \epsilon bvy, \quad \dot{v} = uw - \epsilon buy,$$
$$\dot{w} = -uv, \quad \epsilon\dot{x} = -y, \quad \epsilon\dot{y} = x + buv,$$

was derived by Lorenz by truncation of the shallow-water equations (see Lorenz & Krishnamurthy 1987). In the quasi-geostrophic regime, with $\epsilon \ll 1$ and $b = O(1)$, it is easy to derive equations for the slow manifolds $\mathcal{M}_n$; these are given by the (asymptotic but divergent) series

$$\begin{aligned} x &= -buv + \epsilon^2 b(uv^3 - u^3v - 4uvw^2) + \cdots, \\ y &= \epsilon b(u^2 - v^2)w + \cdots. \end{aligned}$$

Truncating these expansions optimally, one can define the balanced contribution $(x_{\mathsf{bal}}, y_{\mathsf{bal}})$ to (x, y) and, by subtraction, the IGW contribution, expected to be of the form

$$(x_{\mathsf{igw}}, y_{\mathsf{igw}}) \approx C(\cos(t/\epsilon + \phi), \sin(t/\epsilon + \phi))$$

for some C and ϕ.

Now, the evolution of a balanced initial condition, with $C = 0$, leads to the generation of IGWs and thus to an exponentially small $C > 0$. This is exemplified in Fig. 1 for a particular balanced initial condition near the homoclinic solution that exists for (u, v, w) when $\epsilon = 0$. The change in C occurs abruptly, in an $\epsilon^{1/2}$ neighbourhood of the intersection of the real t-axis with a Stokes line joining complex conjugates poles of (u, v, w) in the complex t-plane. (For the solution of Fig. 1, these are

* *Corresponding author address:* J. Vanneste, School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK; email: J.Vanneste@ed.ac.uk

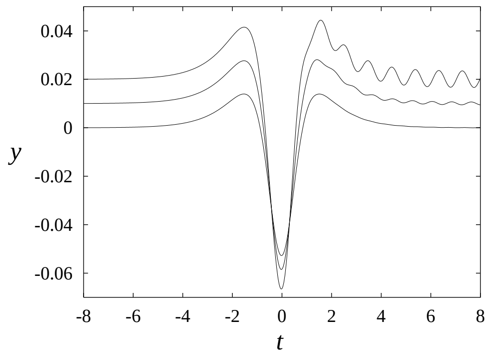

Figure 1: Evolution of y for $b = 0.5$ and $\epsilon = 0.1, 0.125$ and 0.15 (successively offset by 0.01).

at $t = \pm \mathrm{i}\pi/2$, so C changes near $t = 0$.) Using exponential asymptotics, we show in Vanneste (2003) that the amplitude of the waves generated is

$$C \sim \epsilon^{-2} f(b) \exp(-\alpha/\epsilon),$$

where α is the distance of the poles of (u, v, w) to the real t-axis, and the nonlinear function $f(b)$ is determined by a convergent recurrence. Numerical experiments confirm this result.

3. SHEARED DISTURBANCES

The sensitivity of exponentially small effects to model details suggests that the wave generation in Lorenz's model could be an artifact of the truncation used for its derivation. Reassurance that this is not the case is provided by joint work with I. Yavneh in which the 3-D Boussinesq equations are considered (Vanneste & Yavneh 2003). Specifically, we examine the evolution of sheared disturbances superimposed to a horizontal Couette flow $\mathbf{u} = (\Sigma y, 0, 0)$. The vertical vorticity of these disturbances has the form

$$\xi(t) \exp[\mathrm{i}(kx + ly + mz)] + \mathrm{c.c.},$$

with $l = -\Sigma k t$ as a result of the shear. The complex amplitude $\xi(t)$ satisfies a linear second-order inhomogeneous equations.

As in Lorenz's model, a balanced contribution to $\xi(t)$ can be defined using an optimally truncated asymptotic series with $\epsilon = |\Sigma|/f$ as small parameter; the remainder takes the form of waves, with

$$\xi_{\mathsf{igw}}(t) \sim C \exp(\mathrm{i}\omega t/\epsilon + \phi),$$

where ω/ϵ is the IGW frequency. Exponential asymptotics indicates that a (balanced) solution with $C = 0$ for $t < 0$ results in

$$C \sim \epsilon^{-1/2} \beta \exp(-\alpha/\epsilon) > 0$$

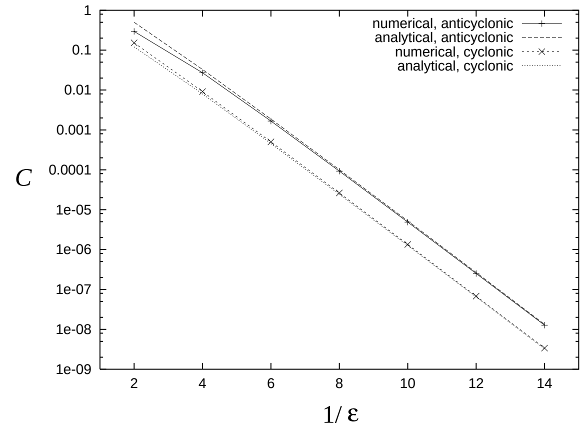

Figure 2: IGW amplitude C as a function of $1/\epsilon$ for sheared disturbances.

for $t > 0$. Here, α and β are given explicitly in terms of elliptic functions of m/k and f/N. A comparison between this estimate and the results of numerical experiments in shown in Figure 2. Interestingly, the wave generation is stronger (by an $O(1)$ factor) for anticyclonic shear than for cyclonic shear.

4. CONCLUSIONS

A crucial factor for the generation of IGW is the location of singularities of the balanced motion in the complex t-plane. When these lie at some finite distance of the real t-axis, the IGWs are exponentially small in ϵ. Future work will examine whether this conclusion, which has emerged from the study of low-order models, also holds for partial differential equations.

REFERENCES

Bokhove, O. & Shepherd, T. G. 1996: On Hamiltonian balanced dynamics and the slowest invariant manifold, *J. Atmos. Sci.* **53**, 276–297.

Camassa, R. & Tin, S.-K. 1996: The global geometry of the slow manifold in the Lorenz–Krishnamurthy model, *J. Atmos. Sci.* **53**, 3251–3264.

Lorenz, E. N. & Krishnamurthy, V. 1987: On the nonexistence of a slow manifold, *J. Atmos. Sci.* **44**, 2940–2950.

Vanneste, J. 2003: Inertia-gravity-wave generation by balanced motion: revisiting the Lorenz–Krishnamurthy model, submitted to *J. Atmos. Sci.*

Vanneste, J. & Yavneh, I. 2003: Exponentially small inertia-gravity waves and the breakdown of quasi-geostrophic balance, submitted to *J. Atmos. Sci.*