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ENERGETICS OF A SYMMETRIC CIRCULATION WITH MOMENTUM CONSTRAINTS

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1. INTRODUCTION

The diagnostic of the energy cycle in the atmosphere is based on the concept of available potential energy (APE). The classical theory of APE (Lorenz 1955) accounts only for thermal constraints, but not for momentum constraints. However, there are cases where the motion is usefully constrained by momentum or angular momentum conservation (*e.g.* quasi-steady symmetric circulations, such as the zonal–mean flow in the atmosphere and hurricanes), for which the classical theory gives an overestimate of the APE and, under certain conditions, fails to correctly diagnose the circulation from a causal point of view.

In the present study a revised theory of APE for symmetric circulations is developed. Our approach is based on the concept of pseudoenergy, which arises from the underlying Hamiltonian structure of the equations of atmospheric fluid dynamics (Shepherd 1990). The concept of APE is generalized to a non-resting reference state, thereby incorporating momentum constraints. The theory is presented in detail in Codoban and Shepherd (2003).

2. BACKGROUND

2.1 *Governing equations*

We work out the theory for the case of the non-hydrostatic, f-plane Boussinesq equations, to illustrate the approach. We consider the x-averaged equations, with no explicit dissipation, and treat the eddy fluxes as forcings. By the non-divergence property of the meridional flow we introduce a streamfunction ψ in the y-z plane such that $v = -\psi_z, w = \psi_y$. By writing the equations in the Transformed Eulerian Mean (TEM) format the forcing in the thermodynamic equation consists only of the diabatic heating R, the forcing in the x-momentum equation is the EP flux divergence X, and ψ represents the TEM circulation.

We analyse the system with respect to a time-independent reference state (RS) defined by

$$\psi \equiv 0, \quad m = M(y,z), \quad \theta = \Theta(y,z), \tag{1}$$

and obeying thermal-wind balance $M_z = -(g/f\theta_0)\Theta_y$. Here $m \equiv u - fy$ is the absolute momentum and θ is the departure of potential temperature from a constant basic state value θ_0. At finite amplitude the system reads

$$m'_t + \partial(\psi', M + m') = X, \tag{2}$$

$$v'_t + \partial(\psi', v') = -p'_y - fm', \tag{3}$$

$$w'_t + \partial(\psi', w') = -p'_z + (g/\theta_0)\,\theta', \tag{4}$$

$$\theta'_t + \partial(\psi', \Theta + \theta') = R. \tag{5}$$

Here $m', \theta', v', w', \psi'$ are the disturbances around the corresponding RS values, $p' \equiv p^* - p^*_{\rm RS}$, with $p^* \equiv p/\rho_0 + f^2y^2/2$, the subscripts denote partial derivatives, $\partial(g,h) \equiv g_y h_z - g_z h_y$ is the two-dimensional Jacobian operator, and we neglect for now the eddy forcings in (3) and (4).

2.2 *Available energy*

In the conservative case, when $X = 0 = R$, the Hamiltonian of the system (2)–(5) is given by

$$\mathcal{H} = \iint_D \left\{ \frac{1}{2}|\nabla\psi|^2 + mfy - \frac{\theta g z}{\theta_0} \right\} dydz, \tag{6}$$

with $\psi = 0$ at the boundaries (Cho et al. 1993). Beside the energy itself, the system also has a class of Casimir invariants of the form

$$\mathcal{C} = \iint_D C(m,\theta)\, dydz. \tag{7}$$

The arbitrary function $C(\cdot\,,\cdot)$ has to be chosen in such a way that the conserved quantity $\mathcal{H}+\mathcal{C}$ defines a positive-definite measure of disturbance energy relative to the RS (1). Hence, the RS must be a conditional extremum for $\mathcal{H}+\mathcal{C}$, which requires the functional derivatives to obey

$$\frac{\delta\mathcal{H}}{\delta m} = -\frac{\delta\mathcal{C}}{\delta m}, \quad \frac{\delta\mathcal{H}}{\delta\theta} = -\frac{\delta\mathcal{C}}{\delta\theta}, \tag{8}$$

when evaluated at the RS. The pseudoenergy is given by

$$\mathcal{A} = \mathcal{H} + \mathcal{C} - \mathcal{H}^{\rm RS} - \mathcal{C}^{\rm RS}, \tag{9}$$

with $\mathcal{H}^{\rm RS}$ and $\mathcal{C}^{\rm RS}$ the energy and Casimir, respectively, evaluated at the RS. The APE is the non-kinetic part of $\mathcal{A}$ (see Shepherd (1993) for details).

3. DIAGNOSTIC THEORY

We first study the semi-geostrophic approximation of a nearly symmetric balanced flow, by which (3) is replaced by geostrophic balance, (4) by hydrostatic balance, and the disturbances obey the thermal-wind balance

$$m'_z = -(g/f\theta_0)\,\theta'_y. \tag{10}$$

It follows that the kinetic energy density (KE) of the meridional circulation, $\frac{1}{2}|\nabla\psi|^2$, is neglected.

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3.1 ***Small-amplitude analysis***

The governing equations consist of (2), (5) (with m', θ' dropped from the lhs Jacobians) and (10). The small-amplitude theory is valid if $\mathrm{Ro} \equiv u'/f\ell \ll 1$ (ℓ is a characteristic horizontal length scale), where Ro is defined by u', not by u itself.

In this case, the APE density is given by

$$APE = \frac{1}{2}\left\{C_{mm}(m')^2 + 2C_{m\theta}m'\theta' + C_{\theta\theta}(\theta')^2\right\}, \quad (11)$$

where $C_{(\cdot,\cdot)}$ are the second derivatives of the Casimir density evaluated at the RS. For $X = 0 = R$ the APE is conserved, while for $X \neq 0$ and $R \neq 0$ we get

$$\frac{d}{dt}\iint_D APE\, dydz = \underbrace{\iint_D \left(C_{mm}m' + C_{m\theta}\theta'\right) X\, dydz}_{\mathcal{S}_X}$$
$$+ \underbrace{\iint_D \left(C_{m\theta}m' + C_{\theta\theta}\theta'\right) R\, dydz}_{\mathcal{S}_R} \equiv \mathcal{S}, \quad (12)$$

where $\mathcal{S}_X$ ($\mathcal{S}_R$) is the mechanical (thermal) source (or sink) of APE, respectively. The local form of (12) is

$$(APE)_t + \left((g/\theta_0)\theta'\psi'\right)_y + \left(fm'\psi'\right)_z = S, \quad (13)$$

where S is the density of $\mathcal{S}$.

3.2 ***Finite amplitude and inclusion of inertia terms***

At finite amplitude the system consists of (2), (5) and (10). The density of APE is given by

$$APE = C - C^{\mathrm{RS}} - C_m^{\mathrm{RS}} m' - C_\theta^{\mathrm{RS}} \theta', \quad (14)$$

and the local form of its time tendency equation is

$$(APE)_t + \partial(\psi', APE) + \left((g/\theta_0)\theta'\psi'\right)_y + \left(fm'\psi'\right)_z = S, \quad (15)$$

where

$$S = (C_m - C_m^{\mathrm{RS}})X + (C_\theta - C_\theta^{\mathrm{RS}})R. \quad (16)$$

The semi-geostrophic approximation is valid as long as $\left(u'/f\ell\right)\left(f^2\ell^2/N^2h^2\right)^2 (r/f)^2 \ll 1$, where h is a characteristic vertical scale, and r is the adiabatic damping rate. The second factor is the square of the inverse of the Burger number, which is $O(1)$ in QG scaling.

To include the inertia terms one has to consider the full Hamiltonian (6). The governing equations are (2)–(5), and for the pseudoenergy density A we obtain

$$A = KE + APE = \frac{1}{2}|\nabla\psi'|^2 + C - C^{\mathrm{RS}} - C_m^{\mathrm{RS}} m' - C_\theta^{\mathrm{RS}} \theta'. \quad (17)$$

For the time tendency equations of APE and KE we get

$$(APE)_t + \partial(\psi', APE) = -C_T - C_M + S, \quad (18)$$
$$(KE)_t + \partial(\psi', KE + p') = C_T + C_M, \quad (19)$$

where S is given by (16) and

$$C_M = -fm'v', \quad C_T = (g/\theta_0)\theta' w' \quad (20)$$

are the $KE \leftrightarrow APE$ conversion terms. Thus the inclusion of the inertia terms does not change the source/sink terms in the energetics, but introduces a kinetic energy component (of the flow in the y-z plane) with thermal (C_T) and mechanical (C_M) conversion terms between the kinetic and available potential energy.

4. APPLICATION

We consider the case of a symmetric zonal flow with a negative zonal force driving a positive meridional flow. The diabatic heating is given by the Newtonian cooling approximation $R = -r(\theta - \theta_{\mathrm{rad}})$, where the radiative equilibrium temperature profile is

$$\theta_{\mathrm{rad}}(y, z) = -\left(f\theta_0\lambda/g\right)y + \left(N^2\theta_0/g\right)z. \quad (21)$$

From the point of view of causality this is a mechanically driven circulation with thermal damping.

We analyze the circulation in the small-amplitude approximation, with respect to two different RS. For a non-resting RS we take $M(y,z) = \lambda z - fy$, $\Theta = \theta_{\mathrm{rad}}$. For a resting RS we take instead $U = 0$, $\Theta = \left(N^2\theta_0/g\right)z$, so that $M(y) = -fy$. The forcing is given by

$$X = \alpha\left[f\frac{L}{H}\cos\left(\frac{\pi z}{H}\right)\cos\left(\frac{\pi y}{L}\right) - \lambda\sin\left(\frac{\pi z}{H}\right)\sin\left(\frac{\pi y}{L}\right)\right], \quad (22)$$

in the domain $0 \leq z \leq H, -L/2 \leq y \leq L/2$. One has to take $\lambda = 0$ in (22) in the case of the resting RS, in order to get the same solution for the streamfunction.

For the non-resting RS we find that in a steady state

$$\mathcal{S}_X = \frac{\alpha^2 HL}{4r(N^2 - \lambda^2)}\left(N^4 + 2N^2\lambda^2 + f^2\lambda^2\frac{L^2}{H^2}\right). \quad (23)$$

We see that $\mathcal{S}_X > 0$ (since $N^2 > \lambda^2$, as follows from the criteria of symmetric stability) so the circulation is always diagnosed as mechanically driven and thermally damped.

With the resting RS (Lorenz theory) we find instead

$$\mathcal{S}_X = HL\left(\frac{N^2\alpha^2}{4r} - \frac{4\alpha f\lambda L}{\pi^3}\right), \quad (24)$$

which shows that the circulation is diagnosed as thermally driven and mechanically damped for sufficiently small α (i.e. for sufficiently weak forcing), which contradicts causality.

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