3.4

Physical Basis for Empirical PV-streamfunction Spectral Relationships

Christos M. Mitas* and Walter A. Robinson

1. Introduction

It is generally assumed that the barotropic vorticity equation applied to an upper tropospheric level can describe the low-frequency behavior of the atmosphere. Quantitative agreement with observations has, however, been lacking. In an effort to improve the quantitative agreement we developed an empirically modified barotropic model that includes baroclinic effects by using the conservation of potential vorticity (PV, q) instead of absolute vorticity. The modification of the barotropic operator amounts to estimating the spectral relationship between PV and streamfunction (ψ) by linear regression, and using this empirical relation instead of the more familiar barotropic squared wavenumber.

In this paper, we explore some theoretical underpinnings for the empirical relationship used in our model. We use the quasi-geostrophic (QG) framework as the basis for these calculations. First, we solve the Green's function problem for PV in the interior of a QG troposphere. In addition, we solve the same problem but for PV located on the tropopause, with a QG troposphere below and a more stable QG stratosphere above.

In both cases, the free parameters of the problem can be adjusted so that the empirical spectral $q - \psi$ relationship may be largely recovered. It is argued that the empirical operator may be considered as one of these well understood QG models.

2. Spectral operators

Using the conservation of Rossby-Ertel potential vorticity in a frictionless, adiabatic atmosphere, a generalized model may be constructed that applies at a single atmospheric level:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, q = L\psi,$$

where ψ is the streamfunction, J is the Jacobian operator, and L is a linear operator. For the standard barotropic model $L = \nabla^2$.

* *Corresponding author address:* Christos M. Mitas, Univ. of Illinois at Urbana-Champaign, Dept. of Atmospheric Sciences, Urbana, IL 61801; e-mail: mitas@atmos.uiuc.edu

To estimate the spectral relationship between potential vorticity and streamfunction, data from the NCEP reanalysis is used. An effective squared wavenumber can be calculated by performing a linear regression of each spherical harmonic component of the isentropic potential vorticity against the same component of the streamfunction. We may eliminate the dependence on the zonal wavenumber (m) by taking the average along constant total wavenumber (n).

Figure 1 presents the barotropic squared wavenumber ($n(n+1)$), the one dimensional empirical squared wavenumber (i.e. depending only on n), and its quadratic fit. Notice the lower (higher) values of the empirical squared wavenumber with respect to the barotropic one for n smaller (larger) than about 9.

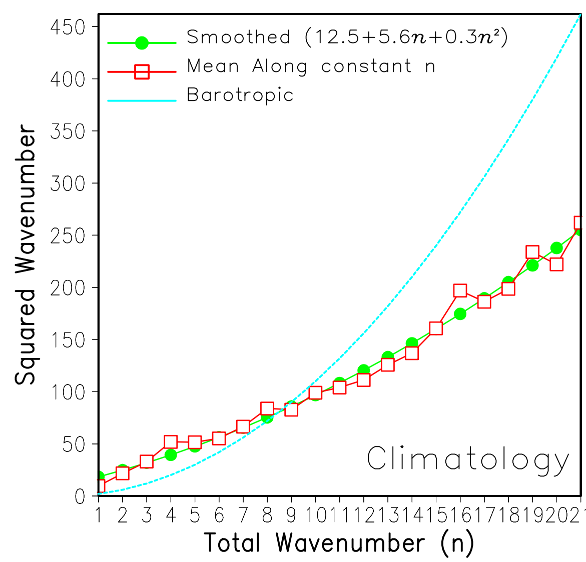

Figure 1: Barotropic and effective squared wavenumber as a function of only the total wavenumber (n), for the winter (DJF) of the NCEP climatological year on the $345\,\mathrm{K}$ isentropic level.

3. QG PV inversion in the tropospheric interior

The basic equation of the quasi-geostrophic (QG) framework using geometric height is given in White (1977) as the conservation of perturbation potential vorticity:

$$q = \nabla^2\psi + \frac{f^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial\psi}{\partial z}\right), \qquad (1)$$

where $\psi[\mathrm{m}^2\ \mathrm{s}^{-1}]$ is the perturbation streamfunction, $f[\mathrm{s}^{-1}]$ is the constant Coriolis parameter, $\rho_0[\mathrm{kg\ m}^{-3}]$ is the standard atmosphere's density which is only a function of height z, and $N[\mathrm{s}^{-1}]$ is the Brunt-Väisälä frequency. A variety of boundary conditions for ψ may be used to solve the PV equation (1). The one we choose for the present section assumes that the streamfunction is bounded at the upper boundary and that there is no perturbation potential temperature at the lower one.

The hydrostatic relation may be written in terms of the potential temperature and streamfunction as:

$$g\frac{\Theta'}{\Theta_0} = f\left(\frac{\partial\psi}{\partial z} - \frac{\psi}{H_\Theta}\right), \qquad (2)$$

where g is the gravity ($9.81\,\mathrm{m\ s}^{-2}$); $\Theta = \Theta_0 + \Theta'$ is the potential temperature separated in a basic state that only depends on height: $\Theta_0(z) = \Theta_{00}e^{z/H_\Theta}$, and the perturbation potential temperature Θ'; $H_\Theta^{-1} = d\log\Theta_0/dz$ is the scale height of the potential temperature, and $\Theta_{00} = \Theta_0(z=0)$.

We use the Earth's radius (a) and the inverse of the Earth's rotation (Ω^{-1}) as our length and time units, respectively. We also define the density scale height as: $H_\rho^{-1} = d\log\rho_0/dz$ or $\rho_0(z) = \rho_{00}e^{-z/H_\rho}$, where $\rho_{00} = \rho_0(z=0)$. Furthermore, we focus on the case of zero perturbation potential temperature at the ground ($\Theta' = 0$), since we try to investigate the simplest possible dynamical situation. Thus, equations (1) and (2) become the set of equations that we solve:

$$q = \nabla^2\psi + (f/N)^2\left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{H_\rho}\frac{\partial\psi}{\partial z}\right) \qquad (3)$$

$$\frac{\partial\psi}{\partial z} - \frac{1}{H_\Theta}\psi = 0 \quad \text{at} \quad z = 0 \qquad (4)$$

Our objective is to find a streamfunction solution by solving the PV equation (3) under the boundary conditions of (4), given a distribution of PV that is a delta function in the vertical and a sinusoidal in the horizontal. In other words we have: $q(x,z) = \delta(z-\xi)e^{ikx}$, where ξ is the level at which the delta function is non-zero and k is a non-dimensional wavenumber, and we look for a streamfunction of the form: $\psi(x,z) = y(z)e^{ikx}$. Replacing into equations (3) and (4) we obtain an ordinary differential equation:

$$\delta(z-\xi) = -k^2y(z) + (f/N)^2\left(y_{zz} - (1/H_\rho)y_z\right) \Rightarrow$$

$$y'' - H_\rho^{-1}y' - (fk/N)^2 y = \delta \qquad (5)$$

and its boundary condition:

$$y' - H_\Theta^{-1}y = 0 \text{ at } z = 0 \qquad (6)$$

Note that primes denote differentiation with respect to z, the only variable on which y depends. The solution of equations (5) and (6) may achieved by solving the homogeneous problem and then combining them to find the Green's function:

$$g(z,\xi) = \begin{cases} \left(e^{\lambda_2 z} - \alpha e^{\lambda_1 z}\right)\frac{e^{\lambda_2\xi}}{\alpha(\lambda_1-\lambda_2)e^{(\lambda_1+\lambda_2)\xi}} & z > \xi \\ e^{\lambda_2 z}\frac{e^{\lambda_2\xi}-\alpha e^{\lambda_1\xi}}{\alpha(\lambda_1-\lambda_2)e^{(\lambda_1+\lambda_2)\xi}} & z \le \xi \end{cases} \qquad (7)$$

where

$$\lambda_{1,2} = H_\rho^{-1}\left(1 \pm \sqrt{1 + 4\left(NH_\rho/f\right)^2 k^2}\right) \qquad (8)$$

are the roots of the characteristic polynomial of the homogeneous version of equation (5), and

$$\alpha = \frac{\lambda_2 - H_\Theta^{-1}}{\lambda_1 - H_\Theta^{-1}}. \qquad (9)$$

Figure 2 presents the streamfunction response to unit PV concentration on different levels ($\xi = 7, 8, 9\,\mathrm{km}$). The parameters have been chosen as follows: $N = 1.2\times10^{-3}\,\mathrm{s}^{-1}$, $f = 1.2\times10^{-4}\,\mathrm{s}^{-1}$, corresponding to a latitude of $\phi \sim 55\,^\circ$, $H_\Theta = 600\,\mathrm{km}$ and $H_\rho = 10\,\mathrm{km}$.

Our interest focuses at the level of application of the delta PV function, $z = \xi$. At that level, the two branches of the Green's function in (7) are equal and become:

$$g(\xi,\xi) = \frac{e^{2\lambda_2\xi} - \alpha e^{(\lambda_1+\lambda_2)\xi}}{\alpha\left(\lambda_1 - \lambda_2\right)e^{(\lambda_1+\lambda_2)\xi}} \qquad (10)$$

In an attempt to simplify the previous equation we define the following quantities: $L = NH_\rho/f$, $\gamma = \sqrt{1+4L^2k^2}$, $\sigma = H_\rho/\alpha$, where L is the usual Rossby radius of deformation. With these definitions, we have the following relations:

$$\lambda_{1,2} = H_\rho^{-1}(1 \pm \gamma),$$

$$\lambda_1 + \lambda_2 = H_\rho^{-1},$$

$$\lambda_1 - \lambda_2 = H_\rho^{-1}\gamma.$$

After some manipulation, and denoting the Green's function amplitude of the streamfunction response $g(z =$

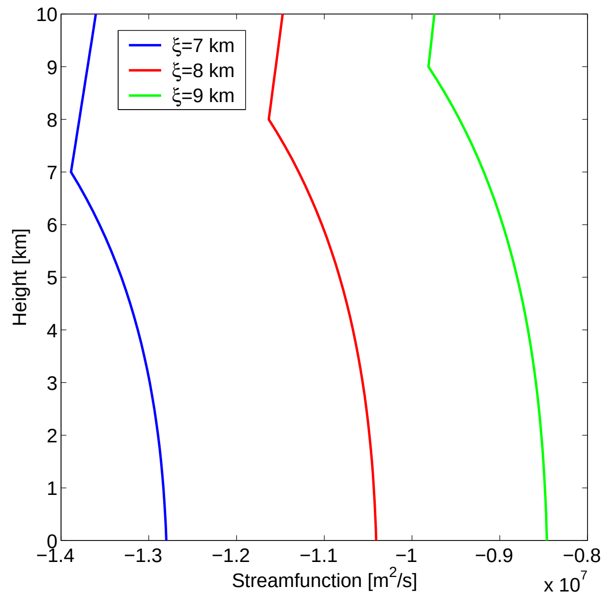

Figure 2: Streamfunction response to PV concentration on different levels. The horizontal wavenumber of the streamfunction is $k = 12$.

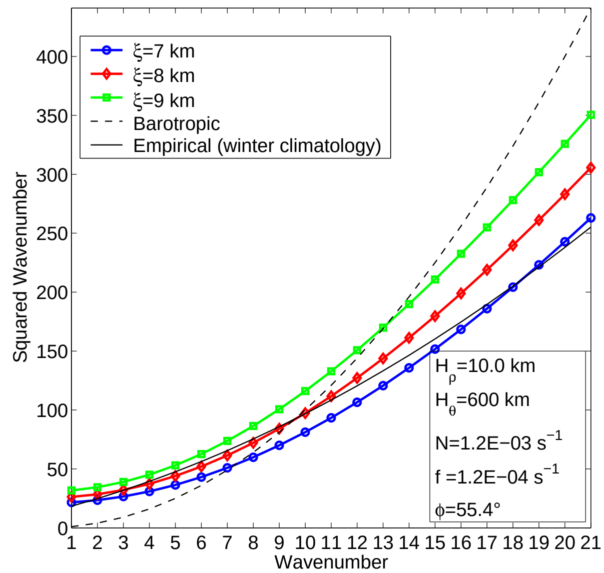

Figure 3: Squared wavenumbers computed with various methods: (a) empirical using NCEP winter climatology, thick line, (b) barotropic, dot-dashed line, (c) interior QG PV inversion at $\xi = 7, 8, 9\,\mathrm{km}$ shown as oval blue line, square red line and diamond green line, respectively. The parameters used to compute the squared wavenumbers are also displayed; see text for details.

$\xi, \xi)$ as ψ_k so that the dependence on the wavenumber is explicitly shown, we have the simplified relation:

$$\psi_k = -\frac{H_\rho - \sigma e^{-\gamma\xi/H_\rho}}{\gamma}. \qquad (11)$$

This equation provides an effective squared wavenumber that relates a unit PV concentration to streamfunction response. The wavenumber dependence is implicit in the definition of both σ and γ. Figure 3 presents two effective squared wavenumbers computed using (11) at different levels ($\xi = 7, 8, 9\,\mathrm{km}$) as well as the barotropic squared wavenumber ($n(n+1)$) and the empirical one presented in Figure 1.

In the present QG model of the interior troposphere we limit our tunable parameters to four. Namely, the planetary vorticity f, the Brunt-Väisälä frequency N, and the scale heights for density and potential temperature H_ρ and H_Θ, respectively.

4. Tropopause QG PV inversion

Instead of considering the interior of the troposphere as in Section 3., we may investigate a more complex situation where the perturbation potential vorticity is zero in the troposphere and stratosphere, but presents a delta function at the tropopause. The stratosphere is distinct from the tropopause due to two different values of the static stability parameter $N_{s/t}$, where the subscript denotes either the *s*tratospheric or the *t*ropospheric value. For simplicity, in this section we assume the Boussinesq approximation of constant density. We follow the formalism of Juckes (1994) in most of this section. The boundary conditions are that the streamfunction should be continuous across the tropopause and finite (0 with no loss of generality) at $\pm\infty$. We place the tropopause at $z = 0$ with negative (positive) height denoting positions in the troposphere (stratosphere).

Following the same technique as in the previous section, the vertical component of the streamfunction may be given as:

$$\psi(z) = \begin{cases} \psi_k e^{-(kN_s/f)z} & z > 0 \\ \psi_k e^{+(kN_t/f)z} & z < 0 \end{cases} \qquad (12)$$

We assume that at the tropopause there is a perturbation potential temperature Θ^{tp}, that corresponds to a potential vorticity distribution (Bretherton, 1966) given by:

$$q_{tp} = -\frac{N_s^2 - N_t^2}{N_s^2 N_t^2} f g \frac{\Theta^{tp}}{\Theta_{00}} \delta(z), \qquad (13)$$

where $\delta(z)$ is the delta function at $\xi = 0$. Applying the boundary conditions to (12) we find:

$$\psi_k = g \frac{\Theta_k^{tp}}{\Theta_{00}} \frac{N_s - N_t}{N_s N_t} \frac{1}{k},$$

and using the equation (13) we finally have:

$$q_k = \left(-f\frac{N_s+N_t}{N_sN_t}k\right)\psi_k \qquad (14)$$

We have explicitly kept the k dependence on our variables in view of our objective of finding a relation between PV and streamfunction for a particular Fourier component.

In Figure 4 we present the squared wavenumber computed with equation (14). For comparison we include the squared barotropic and empirical wavenumbers as well as the one computed with the interior QG PV inversion equation (11). Notice that the linear relationship this equation provides cannot capture the quadratic nature of the empirical squared wavenumber for the whole range of total wavenumbers n. Nevertheless, it seems to fit the empirical relationship adequately in the low wavenumber range.

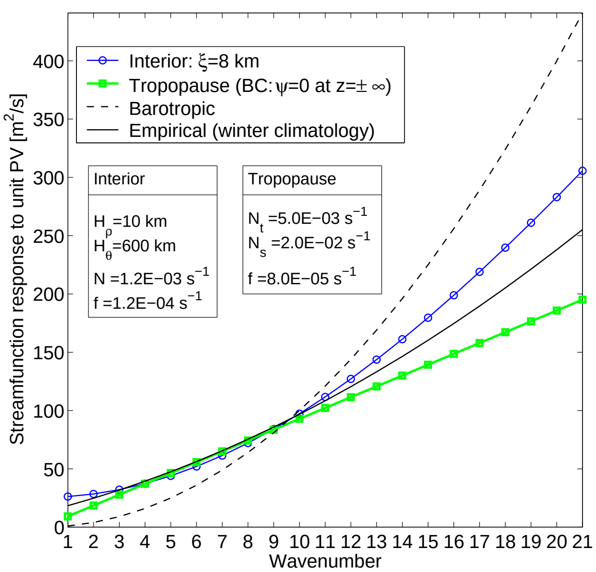

Figure 4: Squared wavenumbers computed with various methods: (a) empirical using NCEP winter climatology, thick line, (b) barotropic, dot-dashed line, (c) interior QG PV inversion at $\xi = 8\,\mathrm{km}$, circle blue line (d) tropopause QG PV inversion, square green line. The parameters used to compute the squared wavenumbers are also displayed; see text for details.

5. Conclusions

The QG models of both the tropospheric interior and tropopause may be used to largely retrieve an empirical $q-\psi$ spectral relationship that has been computed using NCEP reanalysis data. This relationship is different from the barotropic one because of a linear term and a non-zero intercept. Both QG relations have a non-zero intercept, while the tropopause QG relation is linear, thus providing an insight to the linear and constant portion of the empirical relationship ($\underline{a+bn}+cn^2$, see Figure 1 for the plot and values of $a, b,$ and c).

This provides a conceptual framework in which to incorporate our empirical $q-\psi$ spectral relationship. In other words, we may consider some quasi-geostrophic relation between streamfunction and potential vorticity as a proxy of our empirical barotropic model. That gives us significant theoretical background and deeper understanding of the empirical model.

Since the analysis leading to the empirical squared wavenumber is neither on the tropopause nor on a constant height surface, we expect to achieve only adequate rather than identical correspondence to the theoretical result.

6. Acknowledgments

This research is supported by the National Science Foundation, Grant ATM 000 2724.

References

Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteorol. Soc.*, **92**, 325–334.

Juckes, M., 1994: Quasigeostrophic dynamics of the tropopause. *J. Atmos. Sci.*, **51**, 2756–2768.

White, A. A., 1977: Modified quasi-geostrophic equations using geometric height as vertical coordinate. *Quart. J. Roy. Meteorol. Soc.*, **103**, 383–396.