8.5 THE PROPAGATION OF SMALL-SCALE GRAVITY WAVES IN A SLOWLY VARYING BACKGROUND FIELD

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Gravity wave breaking has an important influence on both the momentum and the energy budget of the middle atmosphere. While the role of this process is generally accepted, many details of its development, from initial instabilities over turbulent exchange to a final mixed state, are not clear yet. One difficulty for theoretical studies can be the large range of scales involved. The most prominent gravity waves observed by radar or Lidar measurements have horizontal wavelengths of a few 100 km, while their convective instabilities are typically strongest on scales of a few 10 m, and shear instabilities lead to structures with horizontal scales of a few km. For a complete representation a direct numerical simulation would have to bridge a gap of several orders of magnitude. Given the associated numerical challenge additional methods might be useful. Here an interesting concept is the description of linear wave packets in a slowly-varying background state by a WKB-type approach.

By focussing on the large-scale spatial and temporal dependence of local wave number and amplitude it highlights the essentials of propagating linear disturbances while keeping secondary details aside, thereby enabling an efficient description of small-scale features. Given a set of perturbation variables y_i, e.g. the wind (u, v, and w) and potential temperature θ in the case of a Boussinesq fluid, one expresses them via amplitudes and a phase, i.e.

$$y_i(\mathbf{x},t) = A_i(\mathbf{x},t)e^{i\phi(\mathbf{x},t)} \quad , \tag{1}$$

defines the local wave number $\mathbf{k} = \nabla\phi$ and frequency $\omega = -\partial\phi/\partial t$, and assumes that these two as well as the amplitudes have a slow dependence in space and time, e.g. $|\nabla A_i| \ll |\mathbf{k}A_i|$ and $|\partial A_i/\partial t| \ll |\omega A_i|$. It is then assumed (or deduced from the assumptions above) that local frequency and wave number are connected by a dispersion relation

$$\omega(\mathbf{x},t) = \Omega[(\mathbf{k}(\mathbf{x},t),\mathbf{x},t] \tag{2}$$

depending on the local properties of the time and space dependent background state in which the small-scale structure propagates. For purely kinematical reasons one derives from this the eikonal equations governing the time-dependence of the wave number, i.e.

$$(\frac{\partial}{\partial t} + \mathbf{v_g}\cdot\nabla)\mathbf{k} = -\nabla_\mathbf{x}\Omega \tag{3}$$

where $\mathbf{v_g} = \nabla_\mathbf{k}\Omega$ is the local group velocity.

More difficult is it to obtain an equation for the amplitudes. A first step is to assume that their ratios are given by linear normal mode theory. Then it suffices to determine one unique wave amplitude. In many cases there exits a conserved wave action $\mathcal{A}$ satisfying

$$\frac{\partial\mathcal{A}}{\partial t} + \nabla\cdot(\mathbf{v_g}\mathcal{A}) = 0 \tag{4}$$

which is proportional to the squared wave amplitude. (3) and (4) are used together in ray-tracing algorithms where both wave number and wave action are followed along rays defined by the local group velocity (e.g. Marks and Eckermann, 1995). The roots of this theory in the gravity-wave context go back to a paper by Bretherton (1966) where a wave-activity conservation law has been derived which has later on been generalized further (Bretherton and Garrett, 1968, Bretherton, 1971, Grimshaw, 1975, Müller, 1976, and Andrews and McIntyre, 1978).

Still, however, a general practical theory for the wave amplitude in a background medium with space and time dependence does not exist yet. In Boussinesq fluids wave action conservation seems to be limited to balanced background states with strictly horizontal flow and it applies not to fields with horizontal gradients in the stratification. Also divergent background flows are not included. Especially the first two of these limitations seem to be a problem for a direct application to the propagation of small-scale disturbances in a large-scale gravity wave. Müller (1976) gives an extension of the wave action conservation equation by formulating a corresponding effective background-flow dependent growth or damping rate γ so that

$$\frac{\partial\mathcal{A}}{\partial t} + \nabla\cdot(\mathbf{v_g}\mathcal{A}) = \gamma\mathcal{A} \quad . \tag{5}$$

Even there, however, associated modifications of the dispersion relation are neglected although, as is shown here, they might be important. Moreover, as the present knowledge in the literature seems to be, their exists a clear concept of how to obtain the wave-action conservation equation for balanced background flows. For the more general case, however, a straightforward recipe leading to an equation for the ampli-

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tude development along rays does not exist. So far it was more a trial-and-error approach by which such equations were found.

For these reasons a general WKB-type approach has been developed which works for all systems of coupled linear first-order partial differential equations with coefficients only varying slowly in time and space. It leads directly to suitable ray-tracing equations with the single modification that the wave amplitude equation now becomes

$$\frac{\partial \mathcal{B}}{\partial t} + \nabla \cdot (\mathbf{v_g} \mathcal{B}) = \Gamma \mathcal{B} \quad , \tag{6}$$

where $\mathcal{B}$ is just a quasi-wave action (also proportional to the square of the wave amplitude) which is not conserved in the case of balanced background flows, i.e. even then $\Gamma \neq 0$. However, it can be shown that under the circumstances discussed by Bretherton (1971) for the Boussinesq case (4) can be derived from (6).

Leaving all mathematics aside in this abstract, only an example shall be given of the performance of the theory. It has been applied to the propagation of a small-scale gravity wave packet in a larger-scale wave in a Boussinesq fluid without rotation. By standard spectral methods the two-dimensional $(x-z)$ Boussinesq equations, linearized about a time-dependent gravity wave with horizontal wave length 100 km and vertical wave length 10 km, have been solved in a periodic 500km $\times$ 50km box. The Brunt-Vaisala frequency is 10^{-2} s^{-1}. The initial small-scale wave packet (horizontal wave length 10 km, vertical wave length 2 km) is in the center of the box. The vertical wind of the initial state and that after one period of the background wave (approx. 6300 s) is shown in figure 1. Within this time the packet has, by the interaction with the background wave, split up into three parts. The corresponding amplitudes are shown in figure 2. Using a dispersion relation incorporating the effects of horizontal shear in the basic-wave potential temperature the WKB equations (3) and (6) have also been solved by spectral methods, now however with a much coarser spatial resolution than the complete linear Boussinesq equations (32 $\times$ 32 instead of 256 $\times$ 256). This is the advantage one can gain from the considerably larger spatial scales in amplitudes than in the phase-resolving wave structure. The resulting vertical-wind amplitude after one basic-wave period is shown in figure 3. Clearly, the agreement with the bottom panel of Fig. 2 is quite good. Improvements of the new WKB theory over the one developed by Bretherton (1966, 1971) can be seen in explicit ray-tracing solutions of the equations. A ray is followed which starts right at the center of the wave packet. The vertical-wind amplitude from the wave-resolving model and from the new WKB theory is shown in figure 4. They

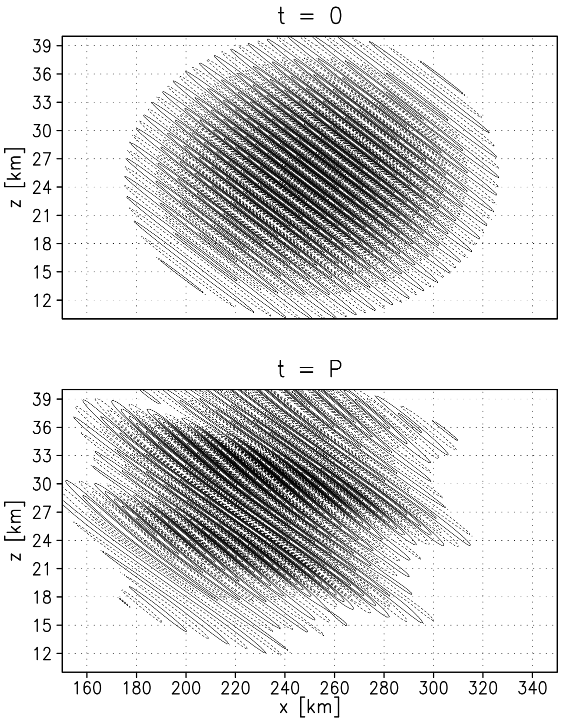

Figure 1: Vertical wind in the initial wave packet (top), and after one period of the background wave. Units are meaningless

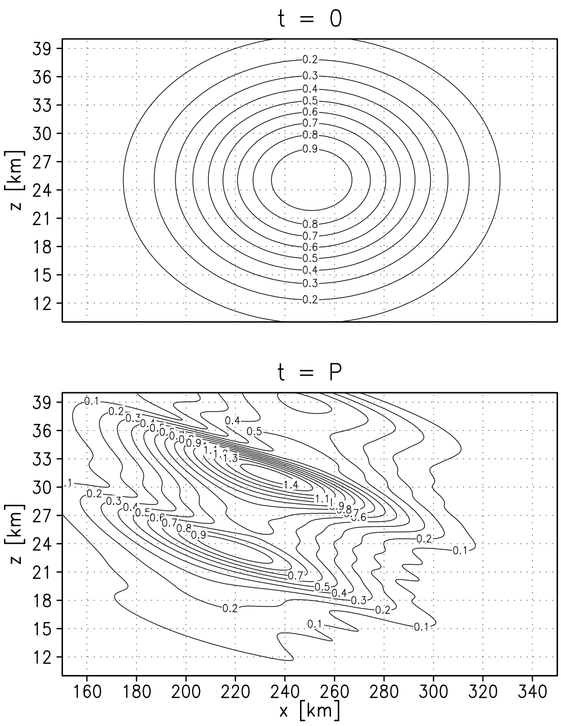

Figure 2: As Fig.1, but now for the amplitude of the vertical wind.

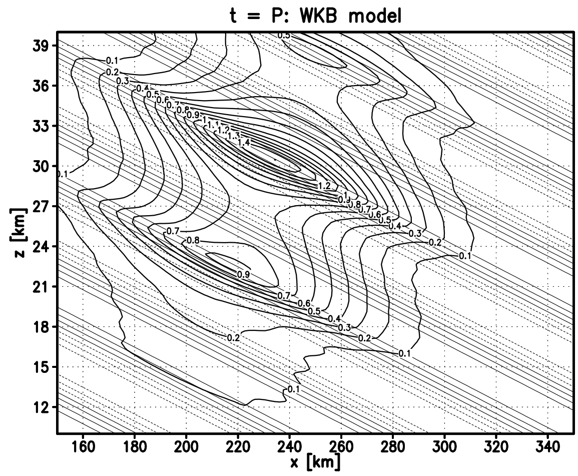

Figure 3: Prediction of the amplitude of the vertical wind in the wave packet by the WKB theory, to be compared to the bottom panel of Fig. 2. Also shown is the zonal wind in the background wave (contour intervals 1 m/s).

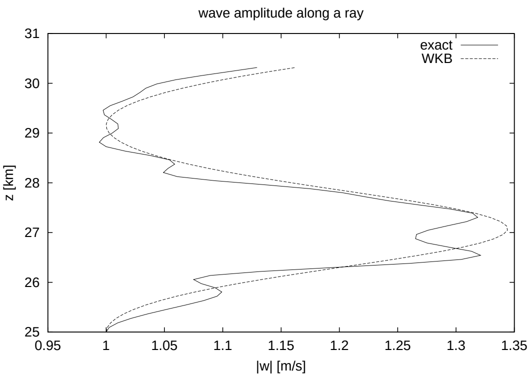

Figure 4: Vertical-wind amplitude along a ray starting at t=0 at the wave packet center, as calculated (1) in the exact wave-resolving model and (2) via the ray-tracing equations from the WKB theory.

are in good agreement. This is to be compared to figure 5 where an analogous result from a calculation with the classical WKB theory is shown. There the simulation of the true amplitude is not so bad either, but definitely worse than in the more general theory.

In summary, a new WKB theory has been developed for the amplitude of small-scale structures in a slowly varying background. It can be shown to be well suitable for the numerically efficient prediction of wave number and amplitudes in convectively stable Boussinesq fluids. Applications to other media should not pose serious problems. Viscosity and heating are incorporated. One future improvement might be to also include these in the dispersion relation. This could help in the treatment of critical

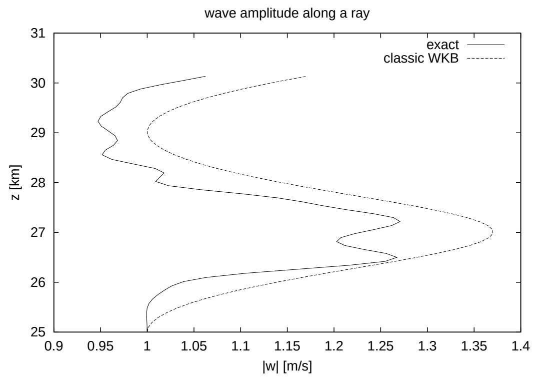

Figure 5: As Fig. 4, but now using the classical WKB ray-tracing equations.

layers which is so far not possible due to the associated singularity in the wave number. An even more challenging problem are convectively unstable situations. Given the gain in efficiency by the WKB approach intensive research in this direction might yet be worthwhile.

References

Andrews, D.G., and M.E. McIntyre, 1978: On wave-action and its relatives. *J. Fluid Mech.*, **89**, 647–664

Bretherton, F.P., 1966: The propagation of groups of internal gravity waves in a shear flow. *Quart. J. Roy. Met. Soc.*, **92**, 466–480

Bretherton, F.P., 1971: The generalized theory of wave propagation. *Lect. Appl. Math.*, **13**, 61–102

Bretherton, F.P., and C.J.R. Garrett, 1968: Wavetrains in inhomogeneous moving media. *Proc. Roy. Soc. London A*, **302**, 529–554

Grimshaw, R., 1975: Internal gravity waves: critical layer absorption in a rotating fluid. *J. Fluid Mech.*, **70**, 287–304

Marks, C.J., and S.D. Eckermann, 1995: A three-dimensional nonhydrostatic ray-tracing model for gravity waves: Formulation and preliminary results for the middle atmosphere. *J. Atmos. Sci.*, **52**, 1959–1984

Müller, P., 1976: On the diffusion of momentum and mass by internal gravity waves. *J. Fluid Mech.*, **77**, 789–823