P1.2 Maximum Entropy Theory and the Decay of 3D Quasi-Geostrophic Turbulence

David A. Schecter*

Department of Earth and Planetary Sciences, Harvard University, Cambridge, MA

1. Introduction and Result

Quasi-geostrophic (QG) turbulence loosely refers to "chaotic" motions within a rapidly-rotating stably-stratified fluid, in particular, an atmosphere or ocean. It is well-known that unforced QG turbulence can decay by coalescence of small-scale eddies into large-scale stationary jets or vortices. In this note, we attempt to predict the 3D end-state of such decay using a form of Lynden-Bell statistical mechanics.[1]

For this study, we ignore viscosity, spatial variation of the Coriolis parameter f, and spatial variation of the buoyancy frequency N. As such, QG turbulence is governed by

$$\partial_t q \ + \ \vec{v} \cdot \nabla q = 0, \quad \vec{v} = \hat{z} \times \nabla \psi, \quad q = \nabla^2 \psi + \partial_{zz} \psi. \tag{1}$$

Here, ∇ is the horizontal gradient operator, t is time, and z is the vertical spatial coordinate, times N/f. At each height z, the potential vorticity q is advected horizontally, without changing value. The (geostrophic) velocity field $\vec{v}$ is obtained from the cross-gradient of the (scaled) dynamical pressure ψ, which acts as a streamfunction. The dynamics is closed by a Poisson equation, which relates ψ to q, and boundary conditions.

Typical simulations of decaying QG turbulence employ spectral[2] or contour dynamics[3] algorithms. Here, we use a particle-in-cell (PIC) code. In particular, the code uses roughly ten-million particles, and a $128 \times 128 \times 65$ x-y-z mesh. The simulation occurs in a unit cube, with periodic boundary conditions in the horizontal coordinates x and y. In addition, $\partial_z \psi = 0$ at $z = 0$ and 1.

The first two columns of Fig. 1 show the beginning and end of a PIC simulation of decaying QG turbulence. At $t = 0$, the flow consists of Gaussian red noise, with mean and r.m.s. potential vorticity (PV) equal to zero and one, respectively. In time, small regions of positive or negative PV chaotically advect, and merge with others of like-sign. Ultimately ($t = 500$), the flow settles into a single pair of vertically aligned, counter-rotating vortices.

We may hypothesize that the initial turbulence of this flow acts to randomly redistribute PV, in a manner that conserves the integral invariants of QG dynamics. There are many conceivable end-states of this random stirring, but we would expect to see the most probable, i.e., the *maximum entropy state* (MES).

Figure 1 juxtaposes the end-state of the simulation to a theoretical MES, described in the following section. The two are in good qualitative agreement, verifying that the observed *coalescence and vertical alignment of like-sign PV is the expected result of random, but constrained, stirring.* However, the simulated vortex cores have higher than "expected" levels of PV. This result might have been anticipated. Many past studies of 2D turbulence also demonstrate the qualitative success of maximum entropy theory, but rarely show perfect agreement.[4] The discrepancy is likely caused by inefficient redistribution of PV within the cores of intense vortices, even during merger.[5]

Note: The author believes that a higher resolution spectral simulation, with sufficiently small hyper-viscosity, would produce a similar discrepancy with the MES. A check is currently planned.

2. Maximum Entropy Theory

The MES in Fig. 1 was calculated using a form of Lynden-Bell statistical mechanics,[1] which has been distinguished from classic point-vortex theory[6] and energy-enstrophy theory.[7] Lynden-Bell theory and its close cousins were used previously to understand the relaxation of turbulence in 2D Euler flow,[8] and in few-layer models of atmospheric and oceanic flow.[9] The following outlines a fully 3D quasi-geostrophic theory, which, to the author's knowledge, has not been published elsewhere.

We first consider a small box centered at a point $\vec{r}$ in the fluid (Fig. 2). This macro-cell contains many micro-columns, which advect the "fine-grained" PV distribution q in the horizontal plane. Let $f(\vec{r}, \sigma)$, times $d\sigma$, denote the fractional volume of a macro-

*Corresponding author address: David A. Schecter, Daly Postdoctoral Fellow, Harvard University, Department of Earth and Planetary Sciences, 20 Oxford St., Cambridge, MA 02138; e-mail: schecter@fas.harvard.edu.

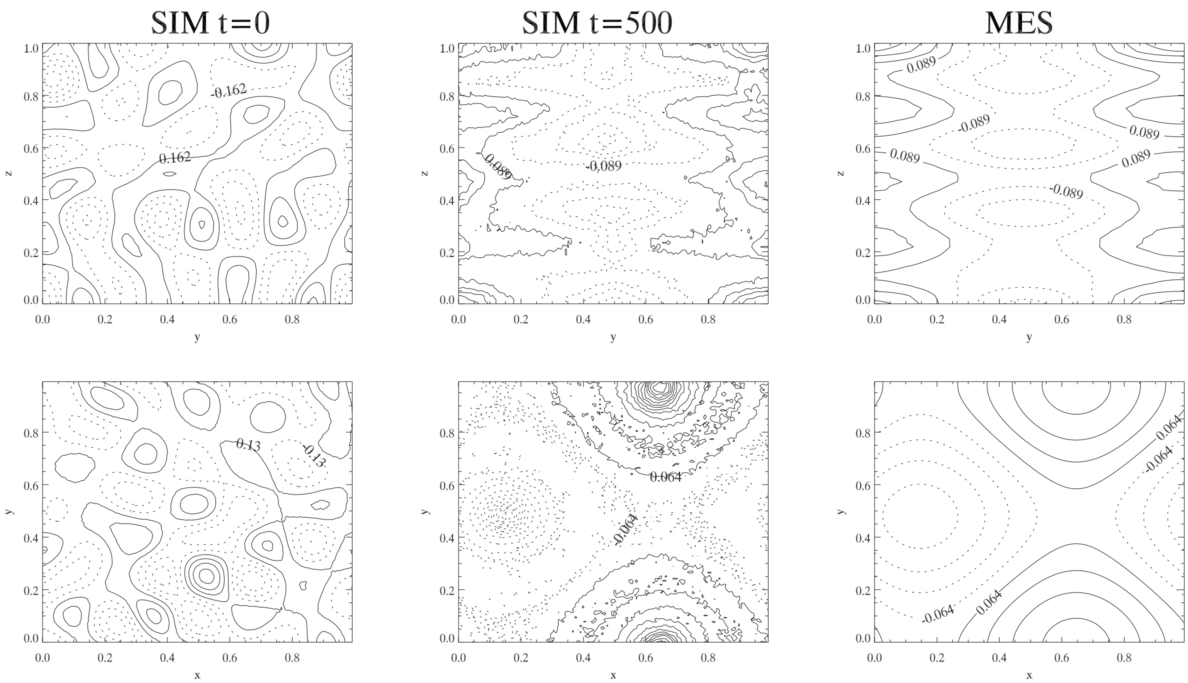

Figure 1: The end-state ($t = 500$) of turbulent decay in a PIC simulation compares favorably to maximum entropy theory. Top: x-averaged PV distributions. Bottom: z-averaged PV distributions. The contours are evenly spaced in PV. Solid/dashed contours indicate positive/negative PV. MES: $\beta = -17.6$, $.41 < |\mu_{\pm}(z)| < .89$.

cell that is filled by micro-columns that carry PV in the range $[\sigma, \sigma + d\sigma]$. The sum over σ of fractional volumes is unity; therefore,

$$F(f;\vec{r}) \equiv \int d\sigma f = 1. \tag{2}$$

We now define the entropy S of the distribution function f. Let S be the logarithm of the number of ways to arrange the micro-columns, within all of the macro-cells, keeping f fixed. By analogy to the entropy of an ideal 2D fluid,[8] we obtain

$$S(f) = -\int dxdydzd\sigma\; f\ln(\sigma_o f), \tag{3}$$

for an ideal 3D QG flow. Equation (3) ignores incidental additive and multiplicative constants, and σ_o is an arbitrary PV, making the argument of the logarithm dimensionless.

The statistically most probable distribution function f is that which maximizes S in a subspace that conserves the invariants of QG dynamics. One such invariant is the total horizontal area, at any height z, that is filled by micro-columns with PV in the range $[\sigma, \sigma + d\sigma]$. This quantity is directly proportional to

$$\Gamma(f;z,\sigma) \equiv \int dxdy\; f. \tag{4}$$

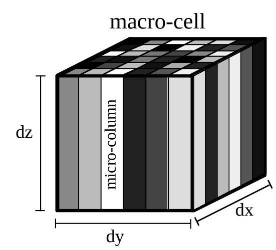

Figure 2: Cartoon of a macro-cell filled with microscopic carriers of "fine-grained" PV (micro-columns). Different shades of grey represent different values of PV.

Another crucial invariant is the energy. Up to a constant factor, the energy is given by

$$E(f) \equiv -\frac{1}{2}\int dxdydz\; \bar{q}\bar{\psi}, \tag{5}$$

in which

$$\nabla^2\bar{\psi} + \partial_{zz}\bar{\psi} = \bar{q} \equiv \int d\sigma\sigma f. \tag{6}$$

Here, $\bar{q}(\vec{r})$ is the PV distribution averaged over a macro-cell, i.e., the "coarse-grained" PV, and $\bar{\psi}(\vec{r})$ is the corresponding streamfunction. We interpret $\bar{q}$ as the observable PV distribution. Conceivably, the conserved energy E has an additional microscopic component. However, it has been shown elsewhere (for the 2D analogue) that this microscopic component is negligible.[8] Henceforth, we will refer to the conserved values of $E(f)$ and $\Gamma(f;z,\sigma)$ as E_o and $\gamma(z,\sigma)$, respectively.

Extrema of S, in the constrained subspace of f, are found by setting equal to zero the first variation of

$$\begin{aligned} S'(f) \equiv \quad & S(f) - \beta E(f) - \int dz d\sigma \mu(z,\sigma)\Gamma(f;z,\sigma) \\ & - \int dxdydz\, \xi(\vec{r})F(f;\vec{r}). \end{aligned} \tag{7}$$

Here, β, $\mu(z,\sigma)$ and $\xi(\vec{r})$ are Lagrange multipliers. The equation $\delta S'/\delta f = 0$, in combination with (2), has a solution at $f = f_*$, where

$$f_*(\vec{r},\sigma) = \frac{e^{\sigma\beta\bar{\psi}_*(\vec{r})-\mu(z,\sigma)}}{\int d\sigma' e^{\sigma'\beta\bar{\psi}_*(\vec{r})-\mu(z,\sigma')}}, \tag{8}$$

and $\bar{\psi}_*$ satisfies

$$\nabla^2\bar{\psi}_* + \partial_{zz}\bar{\psi}_* = \bar{q}_* \equiv \int d\sigma\sigma f_*, \tag{9}$$

with the appropriate boundary conditions. The values of the Lagrange multipliers in (8) are determined by $E(f_*) = E_o$ and $\Gamma(f_*;z,\sigma) = \gamma(z,\sigma)$.

Let $\bar{q}_o(\vec{r})$ denote the form of $\bar{q}$ at $t = 0$. Although this function provides a unique value for E_o, *it does not provide a unique* $\gamma(z,\sigma)$. The values of $\gamma(z,\sigma)$ follow from an additional set of assumptions. The author has tried several, but we here discuss only the simplest. The more complex theories yield nearly the same $\bar{q}_*$, for the example under consideration (Fig. 1).

We will assume that each micro-column has one of only 3 discrete levels of PV: $\sigma_+(z)$, $\sigma_-(z)$ or 0, where $\sigma_+(z)$ and $\sigma_-(z)$ are the maximum and minimum of $\bar{q}_o$ at height z. Then,

$$\gamma(z,\sigma) = \alpha_o(z)\delta(\sigma) + \sum_{j=+,-} \alpha_j(z)\delta[\sigma - \sigma_j(z)], \tag{10}$$

where $\alpha_o = A - \alpha_+ - \alpha_-$, A is the area of the horizontal domain, and δ is a Dirac distribution. If we further assume that the $+$ and $-$ species are initially segregated, then

$$\alpha_\pm(z) = \int dxdy\, H(\pm\bar{q}_o)\bar{q}_o/\sigma_\pm, \tag{11}$$

in which $H(\sigma) = 1, 0$ for $\sigma > 0$, $\sigma < 0$.

In the 3-level model (10), the equation $\Gamma(f_*;z,\sigma) = \gamma(z,\sigma)$ can be satisfied only if

$$e^{-\mu(z,\sigma)} = \delta(\sigma) + \sum_{j=+,-} e^{-\sigma_j(z)\mu_j(z)}\delta[\sigma - \sigma_j(z)]. \tag{12}$$

That is, f_* must have the special form

$$f_*(\vec{r},\sigma) = \frac{\delta(\sigma) + \delta(\sigma-\sigma_+)e^{\sigma_+\Phi_+} + \delta(\sigma-\sigma_-)e^{\sigma_-\Phi_-}}{1 + e^{\sigma_+\Phi_+} + e^{\sigma_-\Phi_-}}, \tag{13}$$

in which $\Phi_\pm(\vec{r}) \equiv \beta\bar{\psi}_*(\vec{r}) - \mu_\pm(z)$. By substituting (13) into the r.h.s. of (9), we obtain

$$\nabla^2\bar{\psi}_* + \partial_{zz}\bar{\psi}_* = \bar{q}_* = \frac{\sigma_+ e^{\sigma_+\Phi_+} + \sigma_- e^{\sigma_-\Phi_-}}{1 + e^{\sigma_+\Phi_+} + e^{\sigma_-\Phi_-}}. \tag{14}$$

The values of E_o and $\alpha_\pm(z)$ determine β and $\mu_\pm(z)$.

To find the Lagrange multipliers, β and $\mu_\pm(z)$, of the MES, we used an iterative scheme that is similar to the 2D algorithm in Turkington and Whitaker.[10] The resulting $\bar{q}_*$ is the right-most column of Fig. 1.

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[1] D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967);

[2] J.C. McWilliams et al., Science **264**, 410 (1994); J. Fluid Mech. **401**, 1 (1999).

[3] D.G. Dritschel and C. Macaskill, Geophys. Astrophys. Fluid Dyn. **92**, 233 (2000).

[4] X.-P. Huang and C.F. Driscoll, Phys. Rev. Lett. **72**, 2187 (1994); D. Marteau et al., Phys. Rev. E. **51**, 5124 (1995); M. R. Brown, J. Plasma Physics, **57**, 203 (1996); P. Chen and M.C. Cross, Phys. Rev. Lett. **77**, 4174 (1996); W.H. Schubert et al., J. Atmos. Sci **56**, 1197 (1999); H. Brands et al., Phys. Fluids **11**, 3465 (1999).

[5] J.C. McWilliams et al., J. Fluid Mech **401**, 1 (1999); G.F. Carnevale et al., Phys. Rev. Lett. **66**, 2735 (1991).

[6] L. Onsager, Nuovo Cimento Suppl. **6**, 279 (1949); G. Joyce and D. Montgomery, J. Plasma Phys. **10**, 107 (1973); T.S. Lundgren and Y.B. Pointin, J. Stat. Phys. **17**, 323 (1977).

[7] R. Salmon in *Topics in Ocean Physics, Prc. Intl. Sch. Phys. 'Enrico Fermi'*, Varenna, Italy, 30 (1982). W.J. Merryfield, J. Fluid. Mech. **354**, 345 (1998).

[8] J. Miller, Phys. Rev. Lett. **65**, 2137 (1990); R. Robert and J. Sommeria, J. Fluid Mech. **229**, 291 (1991); J. Miller et al., Phys. Rev. A. **45**, 2328 (1992).

[9] M.T. DiBattista et al., Geophys. Astrophys. Fluid Dyn. **89**, 235 (1998); M.T. DiBattista and A.J. Majda, J. Phys. Ocean. **30**, 1325 (2000); B. Turkington et al., PNAS **98**, 12346 (2001); P.B. Weichman and D.M. Petrich, Phys. Rev. Lett. **86**, 1761 (2001); P.H. Chavanis and J. Sommeria, Phys. Rev. E **65**, 26302 (2002); F. Bouchet and J. Sommeria, J. Fluid Mech. **464**, 165 (2002).

[10] B. Turkington and A. Whitaker, SIAM J. Sci. Comp. **17**, 1414 (1996);