# P2.2 The Symmetrization Rate of a Geophysical Vortex: Extension of Theory to Large Rossby Numbers

David A. Schecter[a]*and Michael T. Montgomery[b]

[a]*Department of Earth and Planetary Sciences, Harvard University, Cambridge, MA*
[b]*Department of Atmospheric Science, Colorado State University, Fort Collins, CO*

## 1 The Problem

Vortices abound in planetary flow. Familiar examples are the Gulf Stream rings, hurricanes and the polar stratospheric vortex. Many vortices exhibit a tendency to become symmetric, that is, vertically aligned, and circular in the horizontal plane. For example, numerical simulations indicate that the vortices of planetary turbulence symmetrize on average [1], although individual vortices may retain some degree of tilt and ellipticity [2]. Furthermore, symmetrization can, in principle, contribute to the intensification of a swirling storm, such as an incipient tropical cyclone [3,4]. Because of its ubiquity, and possible relevance to weather forecasting, we are motivated to study the fundamental physics of vortex symmetrization.

## 2 Background

Early studies of symmetrization focused on ideal 2D vortices, which are in many respects analogous to atmospheric cyclones and ocean eddies. In their seminal paper (1970), Briggs, Daugherty and Levy pointed out that "quasi-modes" can control the rate at which an ideal 2D vortex becomes axisymmetric [5]. Subsequent laboratory experiments have verified the importance of this mechanism [6,7]. Figure 1a provides an example. We note that in this experiment, the vortex is actually a spinning column of electrons, in a Penning-Malmberg trap. By coincidence, the electron vortex obeys dynamics that are isomorphic with the 2D Euler equations.

The experiment begins with an axisymmetric vortex. The unperturbed vorticity and angular velocity, $\bar{\zeta}(r)$ and $\bar{\Omega}(r)$, decrease monotonically with radius $r$. At $t = 0^+$, an elliptical perturbation is applied. Then, in the *core* of the vortex, the elliptical perturbation rotates with a uniform phase-velocity, and decays exponentially with time. Formally, the core vorticity perturbation is given by $\zeta'(r, \varphi, t) \simeq a(t)Z(r)\cos(n\varphi - \omega t)$, where $Z(r)$ gives the radial variation of the wave-amplitude, $n = 2$, $a(t) = a_o e^{\gamma t}$, and $\gamma < 0$. Because this wave is supported by a radial vorticity gradient, it is classified as a Rossby wave.

The critical radius $r_*$ of the vortex-Rossby-wave is defined by the following "resonance condition"

$$\omega/n = \bar{\Omega}(r_*), \tag{1}$$

and generally lies outside the core. In a critical layer about $r_*$, the streamlines form cat's eyes (Fig. 1b). The flow in the critical layer efficiently stirs vorticity, and causes $\zeta'$ to grow there. Because the vorticity perturbation behaves like a damped wave only in the core, but *grows* in a critical layer, it can not be a damped eigenmode of the vortex. For this reason it is called, instead, a *quasi-mode.*

Conservation of wave-activity (pseudo angular-momentum) explains why stirring of vorticity in the critical layer damps the core vortex-Rossby-wave [8]. For 2D Euler flow, the total wave-activity is the following quadratic integral of the vorticity perturbation: $A = -\int\int d\varphi dr r^2(\zeta')^2/(d\bar{\zeta}/dr)$. The minus sign guarantees that $A$ is positive, assuming that $d\bar{\zeta}/dr < 0$ for all $r > 0$.

For brevity, we may ignore nuances and view the wave activity as having two dynamically relevant parts; specifically, $A \simeq A_{core} + A_{c-layer}$, where

$$\begin{aligned} A_{core} &\equiv -\int_0^{2\pi}\int_0^{r_v} d\varphi dr r^2 \frac{(\zeta')^2}{d\bar{\zeta}/dr}, \\ A_{c-layer} &\equiv -\int_0^{2\pi}\int_{r_*-\delta r}^{r_*+\delta r} d\varphi dr r^2 \frac{(\zeta')^2}{d\bar{\zeta}/dr}. \end{aligned} \tag{2}$$

Here, $r_v$ is the radius of the vortex core, within which lives the core vortex-Rossby-wave. In addition, $r_* - \delta r > r_v$. A more precise definition of $\delta r$ is not important for this abstract. With the present

*Corresponding author address: David A. Schecter, Daly Postdoctoral Fellow, Harvard University, Department of Earth and Planetary Sciences, 20 Oxford St., Cambridge, MA 02138; e-mail: schecter@fas.harvard.edu.

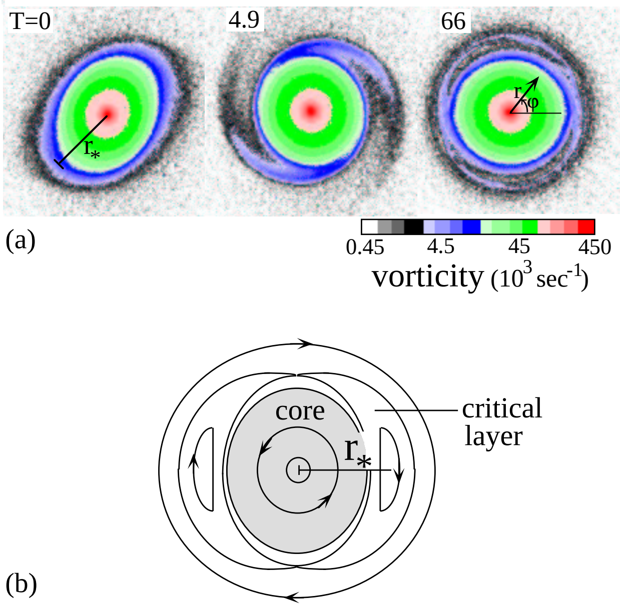


Figure 1: (a) The inviscid axisymmetrization of a 2D vortex. Time $T$ is in units of $2\pi/\bar{\Omega}(0)$. (b) Cartoon of streamlines in a reference frame that co-rotates with the core vortex-Rossby-wave. The core of the vortex is shaded.

decomposition, conservation of wave activity has the form

$$\frac{d}{dt}A_{core} = -\frac{d}{dt}A_{c-layer}. \tag{3}$$

The growth of $(\zeta')^2$ in the neighborhood of $r_*$ causes $A_{c-layer}$ to grow. By (3), this must diminish $A_{core}$, and thereby damp the core vortex-Rossby-wave.

Further (linear) analysis of (3) leads to an equation for the wave amplitude of the form $da/dt = \gamma a$, where

$$\gamma = c\frac{d\bar{\zeta}}{dr}(r_*), \tag{4}$$

and $c$ is a positive factor that depends on the wavenumber $n$, and the specific form of $\bar{\zeta}$. Equation (4) demonstrates that the decay rate of a quasi-mode increases with the vorticity gradient at the critical radius, and is zero if that gradient is zero. The analysis used to derive (4) assumes that $\gamma/\omega << 1$. A more general procedure for calculating the decay rate of a quasi-mode, based on a Laplace transform solution to the initial value problem, is outlined in Briggs, Daugherty and Levy [5].

Three-dimensional symmetrization can also occur by the decay of a quasi-mode. Figure 2 shows a numerical simulation in which a tilted quasi-geostrophic (QG) vortex becomes vertically aligned by this process [4,8]. At $t = 0^+$, the potential vorticity (PV) is misaligned, giving the vortex a tilt. In the core of the vortex, the tilt precesses with a uniform phase-velocity, and decays exponentially with time. Formally, the core PV perturbation is given by $q'(r,\varphi,z,t) \simeq a_o e^{\gamma t} Q(r)\cos(m\pi z/H)\cos(n\varphi - \omega t)$, where $(m,n) = (1,1)$, $\gamma < 0$, $H$ is the vortex height, and $Q(r)$ accounts for the radial variation of the wave-amplitude. This 3D vortex-Rossby-wave is damped by the stirring of PV in its critical layer. Analogous to (4), the decay rate $\gamma$ is proportional to the PV gradient at the critical radius $r_*$.

## 3 Recent Results

At this conference, we will report recent progress on the theory of quasi-modes, and further illustrate their importance to vortex symmetrization. Specifically, we will consider the 3D quasi-modes of a barotropic vortex on the $f$-plane, whose vertical vorticity $\bar{\zeta}$ decreases monotonically with $r$. We will assume that the ambient buoyancy frequency $N$ is constant, and that vertical motion is inhibited at the top ($z = H$) and bottom ($z = 0$) boundaries. We may characterize this basic state, in part, with two dimensionless parameters: the central Rossby number,

$$Ro \equiv \frac{\bar{\zeta}(0)}{f}, \tag{5}$$

and the normalized, ambient, internal Rossby deformation radius

$$\frac{l_R}{r_v} \equiv \frac{NH}{\pi|f|r_v}. \tag{6}$$

Here, $f$ is the constant Coriolis parameter, and $r_v$ is the radial length-scale of the vortex core.

### *3.1 Extension to Rapidly Rotating Vortices*

Previously, we presented the QG theory of 3D quasi-modes [8]. QG theory requires that the Rossby

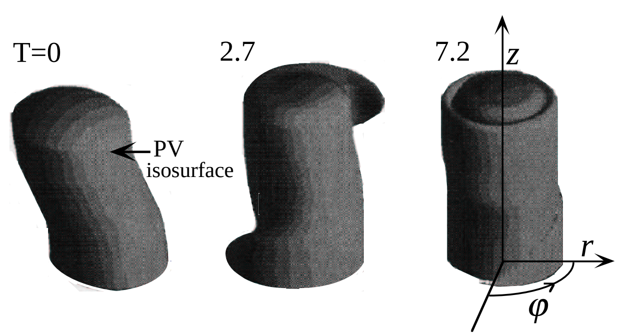


Figure 2: The conservative vertical alignment of a (quasi-geostrophic) vortex. Time $T$ is in units of $2\pi/\bar{\Omega}(0)$.

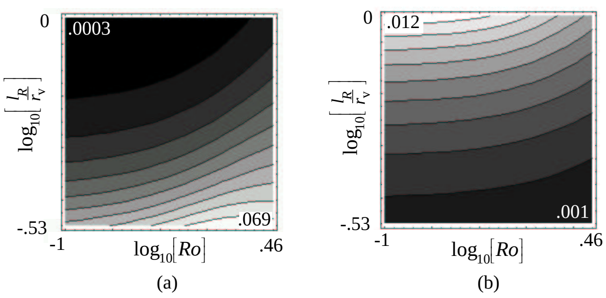


Figure 3: Contour plots of the normalized decay rate, $|\gamma/\bar{\zeta}(0)|$, of the $(m,n) = (1,1)$ quasi-mode of a Gaussian cyclone (a), and a Rankine-with-skirt cyclone (b). The decay rates are here calculated using the asymmetric balance approximation. The printed numbers inside the graphs are the decay rates at $(Ro, l_R/r_v) = (.1, 1)$ and $(2.93, .293)$. Darker shades represent smaller decay rates.

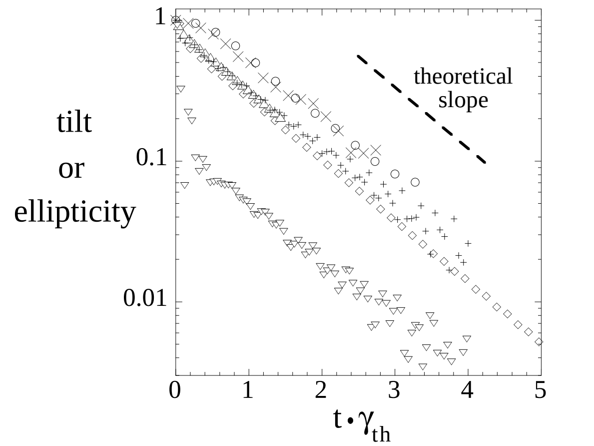


Figure 4: Asymmetry (tilt or ellipticity) versus time for a selection of numerical and laboratory experiments, covering a broad region of parameter space. All asymmetries are normalized to their initial values. See Table 1 for a full description of the data.

number is small, i.e., that $Ro << 1$. Here, we extend the theory to rapidly rotating vortices, which can have Rossby numbers greater than unity. We first generalize in the context of the mathematically elegant, asymmetric balance (AB) model [9,10]. This model formally requires that

$$D_N^2 \equiv \frac{[\omega - n\bar{\Omega}(r)]^2}{N^2} << 1, \tag{7}$$

and furthermore that

$$D_I^2 \equiv \frac{[\omega - n\bar{\Omega}(r)]^2}{\bar{\eta}(r)\bar{\xi}(r)} << 1, \tag{8}$$

where $\bar{\eta} \equiv f + \bar{\zeta}(r)$ is the absolute vertical vorticity and $\bar{\xi}(r) \equiv f + 2\bar{\Omega}(r)$ is the modified Coriolis parameter. The condition $D_N^2 << 1$ amounts to hydrostatic balance. The condition $D_I^2 << 1$ can be satisfied even if $Ro \gtrsim 1$, since the numerator is the square of the *Doppler shifted* quasi-mode frequency.

As in QG theory, AB theory predicts that the decay rate $\gamma$ of a quasi-mode of a barotropic vortex is proportional to the value of $d\bar{\zeta}/dr$ at the critical radius $r_*$. However, the decay rate (normalized to the central vorticity) can either increase or decrease with the parameters $Ro$ and $l_R$, depending on the global form of $\bar{\zeta}$. To illustrate this point, we compare the $(m,n) = (1,1)$ quasi-modes of a Gaussian cyclone and a Rankine-with-skirt (RWS) cyclone. Here, $m$ and $n$ are the vertical and azimuthal wave-numbers. Decay of the $(1,1)$ quasi-mode corresponds to the alignment of a tilted vortex.

The vertical vorticity of a Gaussian cyclone is given by

$$\bar{\zeta} = Z_o e^{-9r^2/2r_v^2}, \tag{9}$$

where $Z_o$ is the central vorticity. The vertical vorticity of an RWS cyclone is given by

$$\bar{\zeta} \simeq \begin{cases} Z_o & r < r_v \\ Z_o\varepsilon(1 - \frac{r}{\alpha r_v}) & r_v < r < \alpha r_v \\ 0 & r > \alpha r_v, \end{cases} \tag{10}$$

where $\varepsilon << 1$ and $\alpha > 1$. This vortex has a uniform core, of radius $r_v$, and an outer skirt of relatively weak vorticity, that decreases linearly with radius to zero at $r = \alpha r_v$.

Figure 3a is a contour plot of the normalized decay rate $|\gamma/\bar{\zeta}(0)|$ of the $(1,1)$ quasi-mode of a Gaussian cyclone, in a small region of the $Ro$-$l_R$ parameter space. In this case, the normalized decay rate *increases* as the Rossby number $Ro$ increases, and as the deformation radius $l_R$ decreases. Figure 3b is a similar plot for the $(1,1)$ quasi-mode of an RWS cyclone, with $\varepsilon = 0.16$ and $\alpha = 8/3$. Unlike before, the normalized decay rate *decreases* as the Rossby number increases, and as the deformation radius decreases.

### *3.2 Limitation of Balance*

One potential draw-back of all "balance" models, including QG and AB theory, is their neglect of inertia-buoyancy (IB) waves. At Rossby numbers greater than unity, the vortex-Rossby-wave, composing the core component of a quasi-mode, can resonantly excite an outward propagating IB wave. If the magnitude of $d\bar{\zeta}/dr$ at $r_*$ is above a threshold, this excitation merely retards the damping of the vortex-Rossby-wave. In this case, AB theory can

| symbol | model | vortex-type | $Ro$ | $l_R/r_v$ | (m,n) | $\lvert\gamma_{th}/\zeta(0)\rvert$ | ref. |
|---|---|---|---|---|---|---|---|
| $\circ$ | QG nonlinear | Gaussian | 0 | 0.28 | (1,1) | .048 | [8] |
| $\bigtriangledown$ | AB linear | "hurricane" | 234 | 34.3 | (1,1) | .0038 | [10] |
| $\bigtriangleup$ | PE linear | Gaussian | 0.25 | 0.75 | (1,1) | .0023 | – |
| $\diamond$ | PE linear | Gaussian | 2.0 | 0.75 | (1,1) | .0092 | – |
| $+$ | PE linear | RWS ($\varepsilon = .16$, $\alpha = 2$) | 10 | 1.0 | (1,2) | .0051 | – |
| $\times$ | 2D $e^-$ plasma | experimental | – | – | (0,2) | .030 | [7] |

TABLE 1: Legend for data in Fig. 4. The last column provides references for some of these results.

still provide a good approximation for the decay rate. However, if the magnitude of $d\bar{\zeta}/dr$ at $r_*$ is below the threshold, the IB wave will cause the core vortex-Rossby-wave to grow, and the vortex to become increasingly *asymmetric*. To accurately account for the influence of IB waves, the first author has recently generalized the theory of quasi-modes to a primitive equation (PE) model, which assumes only hydrostatic balance (i.e., $D_N^2 << 1$).

*3.3 Quasi-Modes "Everywhere"*

We conclude by comparing a set of observed vortex symmetrization rates to theoretical quasi-mode decay rates. The observations are primarily from a sample of numerical simulations that span a broad region of parameter space, and are summarized in Table 1. The numerical simulations are of 3 types: a nonlinear QG simulation, a linearized AB simulation, and a linearized PE simulation. In each case, a barotropic vortex is either tilted or given a $z$-dependent elliptical deformation. Specifically, the initial PV perturbation is of the form $q' \propto r^{n-1}(d\bar{\zeta}/dr)\cos(m\pi z/H)\cos(n\varphi)$, where $m = 1$ and $n = 1$ or 2. Further details of the initial perturbations are discussed in Refs. [8,10]. In time, all of the perturbations decay, and the vortices symmetrize.

Figure 4 is a log-linear plot of the asymmetry (tilt or ellipticity) versus time, for all simulations. The asymmetry is measured by the magnitude of the asymmetric pressure (geopotential) perturbation at a fixed radius in the vortex core. Time is multiplied by the theoretical decay rate $\gamma_{th}$ of the least damped quasi-mode, with wave-numbers $(m, n)$ matching those of the initial perturbation. With this scaling, all curves should have the same slope, provided that symmetrization occurs by decay of the quasi-mode. The theoretical slope is indicated by a dashed line in the figure. Evidently, the simulations are in good agreement with theory, after relatively short initial adjustments.

Figure 4 also compares theory to the laboratory experiment of Fig. 1, in which a 2D elliptical vortex tends toward an axisymmetric state. As indicated by this graph, the symmetrization rate of the experimental vortex is equal to the theoretical decay rate of its $n = 2$ quasi-mode. We note that here, the ellipticity is measured by the quadrupole moment of the vorticity perturbation.

In summary, quasi-modes can play an important role in regulating the rate at which a geophysical vortex symmetrizes. We have extended the theory of 3D quasi-modes to rapidly rotating barotropic vortices, using both an asymmetric balance model, and a primitive equation model. According to linear theory, the decay rate $\gamma$ of a quasi-mode increases with the radial gradient of $\bar{\zeta}$ at the critical radius $r_*$ of that quasi-mode. The normalized decay rate $|\gamma/\bar{\zeta}(0)|$ can either increase or decrease with the parameters $Ro$ and $l_R$, depending on the specific form of $\bar{\zeta}$.


[1] J.C. McWilliams, J.B. Weiss and I. Yavneh, *J. Fluid Mech.* **401**, 1 (1999).

[2] D.G. Dritschel, *J. Fluid Mech.* **371**, 141 (1998).

[3] J. Enagonio and M.T. Montgomery, *J. Atmos. Sci.* **44**, 685 (2001); J.D. Moller and M.T. Montgomery, *J. Atmos. Sci.* **56**, 1674 (1999); M.T. Montgomery and J. Enagonio, *J. Atmos. Sci.*, **55**, 3176 (1998).

[4] P.D. Reasor and M.T. Montgomery, *J. Atmos. Sci.* **58**, 2306 (2001);

[5] R. J. Briggs, J. D. Daugherty and R. H. Levy, *Phys. Fluids* **13**, 421 (1970).

[6] S. Pillai and R.W. Gould, *Phys. Rev. Lett.* **73**, 2849 (1994).

[7] D.A. Schecter, D.H.E. Dubin, A.C. Cass, C.F. Driscoll, I.M. Lansky and T.M. O'Neil, *Phys. Fluids* **12**, 2397 (2000).

[8] D.A. Schecter, M.T. Montgomery and P.D. Reasor, *J. Atmos. Sci.* **59**, 150 (2002).

[9] L.J. Shapiro and M.T. Montgomery, *J. Atmos. Sci.* **50**, 3322 (1993).

[10] D.A. Schecter and M.T. Montgomery, *Dyn. Atmos. Oceans*, (in press).