P1A.10 A GENERAL APPROACH TO DOUBLE MOMENT NORMALIZATION OF DROP SIZE DISTRIBUTIONS

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1. INTRODUCTION

Recently, normalization of drop size distributions (DSD) became a tool to study the variability of DSDs in a systematic manner. Sempere-Torres et al. (1994, 1998: refer to as ST) described a normalization procedure based on the basic notion of scaling functions, power law relationships between moments of DSDs. In essence, this procedure states that if drop size is scaled by a factor, its number concentration is also scaled in a predetermined manner. Their proposed formula for the scaling DSD is

$$N(D) = M_i^{\alpha} g(x_1) \qquad (1)$$

where $M_i (= \int N(D) D^i dD)$ is the ith moment of DSDs and x_1 $(= D M_i^{-\beta})$ is the scaled diameter. The normalized function $g(x_1)$ is called the general distribution function. This general expression summarizes all the previously suggested analytical expression of DSDs and clarifies the relation between these expressions and the power-law relationships between moments of the distribution generally used.

Sekhon and Srivastava (1971) and Willis (1984) show the potential of the normalization of DSDs with two parameters, such as the characteristic diameter and number density. But they assume a specific shape of DSDs (exponential and gamma DSDs). Recently, Testud et al. (2001; refer to as TT) expand their idea without any assumption on the functional form of DSDs and have proposed a double-moment normalization:

$$N(D) = N_0^* F(D / D_m) \qquad (2)$$

where D_m is the volume-weighted mean diameter, a particular characteristic diameter that is derived by $D_m = M_4 / M_3$, and N_0^* is defined as $N_0^* = C_T M_3^5 M_4^{-4}$, where C_T is an arbitrary constant that is chosen as $4^4/\Gamma(4)$. Testud *et al.* point out that the remaining scatter around the normalized function is below the noise level of the disdrometric data, suggesting that their two parameters, the third and fourth moments of the DSDs, are sufficient to capture all the discernable variability.

The purpose of this paper is to extend the ST single-moment scaling normalization to a double-moment scaling normalization and establish an explicit relationship between TT and ST approaches. Furthermore, we will show the advantage and disadvantage of both normalization methods and that the scaling normalization of DSDs is a general way of describing DSDs.

2. A GENERAL DOUBLE-MOMENT SCALING DSD

A general form of ST normalization with two moments of DSDs can be derived by re-normalizing $g(x_1)$ with the jth moment of $g(x_1)$:

$$N(D) = M_i^{\frac{j+1}{j-i}} M_j^{\frac{i+1}{i-j}} h(x_2) \text{ with } x_2 = D M_i^{\frac{1}{j-i}} M_j^{\frac{-1}{j-i}} \qquad (3)$$

where x_2 is the scaled diameter and $h(x_2)$ the "second normalized" DSD. In addition, we obtain the general multiple power law relationship among moments of DSDs:

$$M_n = C_{2,n} M_j^{\frac{n-i}{j-i}} M_i^{\frac{j-n}{j-i}}. \qquad (4)$$

Although we use jth moment of $g(x_1)$, the ith and jth moment of DSDs are necessary in the final form. For detailed derivation and overall contents of this paper, see Lee et al. (2003). Interestingly, all scaling exponents disappear and only the orders (*i* and *j*) of the two DSD moments used in the normalization remain in this final form. In addition, we have not assumed any functional form of shape of normalized DSDs that remains free and to be determined from observations.

In the single-moment normalization, a simple power law between any two moments is assumed. The exponent of this power law is the function of scaling exponent β [$M_n = C_{1,n} M_i^{1+(n-i)\beta}$] and the coefficient is the *n*th moment of $g(x_1)$ [$C_{1,n} = \int g(x_1) x_1^n dx_1$]. Similarly, in the double-moment normalization, the coefficient of the multiple power law in (4) is now the *n*th moment of $h(x_2)$ instead of $g(x_1)$. However, the exponent is purely determined by the orders of moments. Therefore, in this approach the role of β and the normalized general function *g* is now played by the two moments of the original DSD used in the normalization. Thus the two moments should jointly contain all the information for the stratification of DSDs.

In addition, the normalization of TT is a particular case of the double-moment scaling normalization (3) in which the moments order 3 and 4 have been selected as the reference variables. The normalization of TT can also be generalized using the generalized characteristic number density $N_0^{'}$ and generalized characteristic diameter $D_m^{'}$, leading to (3):

$$N(D) = N_0^{'} F(D / D_m^{'}) \text{ with } \begin{matrix} N_0^{'} = M_i^{(j+1)/(j-i)} M_j^{(i+1)/(i-j)} \\ D_m^{'} = \left(M_j / M_i\right)^{1/(j-i)} \end{matrix}. \qquad (5)$$

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Thus, normalization by ST and TT are a particular case of (3) and are based on the same concept, *scaling law* of DSDs.

3. *R-Z* RELATIONSHIPS

When we choose $M_i = C_u R$ (~3.67th moment of the DSD) and $M_n = Z$ (6th moment of the DSD) in (4), the following relationship $Z(M_j, R)$ is obtained:

$$Z = \left[C_{2,6} C_u^{\frac{j-6}{j-3.67}} M_j^{\frac{2.33}{j-3.67}} \right] R^{\frac{j-6}{j-3.67}} \quad (6)$$

where C_u is a constant that adjusts the unit. In the single-moment scaling normalization, the coefficient ***a*** of $Z=aR^b$ is the 6th moment of $g(x_1)$ and the exponent ***b*** is related to the scaling exponent β by $b = 1 + 2.33\beta$. In (6), the exponent of *R-Z* relationship depends on the choice of *j*. We can also obtain $Z(N_0^{'}, R)$ and $Z(D_m^{'}, R)$:

$$Z = C_{2,6} C_{2,3.67}^{-1.5} C_u^{1.5} \left(N_0^{'}\right)^{-0.5} R^{1.5} \quad (7)$$

$$Z = C_{2,6} C_{2,3.67}^{-1} C_u \left(D_m^{'}\right)^{2.33} R \quad (8)$$

The explicit exponent in (7) and (8) does not imply any universal value. It depends on the correlation between $N_0^{'}$ and *R* or between $D_m^{'}$ and *R.* The explicit exponent on *R*, *b*=1.5, in (7) is close to the climatological value ($Z=210R^{1.47}$ in Montreal, for example). This implies that the correlation between $N_0^{'}$ and *R* is low when a set of data is taken from a climatological variety of situations. For Marshall-Palmer DSDs, we expect $Z = aR^{1.5}$ since $N_0^{'}$ is a constant and $D_m^{'} \propto R^{0.21}$. For the equilibrium process, the evolution of DSDs is controlled by the number of drops with increasing *R* and the characteristic diameters remains as a constant. Thus, the proportionality between *R* and *Z* is expected. For the strong aggregation between snow particles, $N_0^{'}$ decreases with increasing *R* and the dependence of $D_m^{'}$ on *R* is stronger than that expected in M-P DSDs, that is, the exponent ***f*** of $D_m^{'} = eR^f$ is larger than 0.21. Thus, the exponent of *R-Z* relationship should be larger than 1.5.

4. DATA ANALYSIS

The data used here are composed of 1208 one-minute DSDs (over 20 hours) measured by the optical spectro-pluviometer (OSP: Salles et al. 1998). DSDs are divided into convective and stratiform rain using the presence of bright band (BB) and horizontal gradient of reflectivity obtained from a near-by scanning radar.

a. Compact representation of DSDs

The normalization of the set of these data is shown in Fig. 1 for the single moment (*R*) and Fig. 2 for the double moment (M_i and M_j).

The scaling exponent β is slightly smaller than the value of M-P DSDs, indicating the exponent of $Z=aR^b$ is less than 1.5. The scatter of normalized DSDs in Fig. 1a and the standard deviation in Fig. 1b (vertical bars) are quite large. This shows the limitation of single-moment scaling normalization in terms of compact representation of DSDs. When all DSDs from different physical processes are normalized together, they do not collapse onto one normalized curve. In other words, all the DSD variability cannot be explained by a single parameter.

In the double-moment normalization in Fig. 2a and b, $\overline{h(x_2)}$ is very similar to the ones reported by TT. In general, the scatter drastically decreases compared with the single-moment normalization, illustrating an advantage of double-moment normalization in terms of a compact representation of DSDs. The standard deviation (*SD*: thick vertical bars in Fig.2b) is still larger than that from the statistical noise (lighter vertical bars next to *SD*) derived by assuming Poisson fluctuations due to undersampling. This can be explained by two facts: 1) the possible physical variability that cannot be described by this normalization and 2) underestimation of the statistical noise by the Poisson process.

The degree of the scatter in normalized DSDs is quantified in terms of the standard deviation of fractional error in the estimation of various moments with average normalized DSD $\overline{h(x_2)}$. Fig. 3 shows results from various combinations of two moments used for the normalization. Not surprisingly, when two consecutive moments are used, the error is almost zero for moments close to the ones used for the normalization due to the self-consistency constraints. When the order of the two moments used for the normalization is lower (higher), the error is smaller at lower (higher) moments. The minimum is broader when the order is higher. This simply indicates that the slope of the DSDs has less variability at the larger drop sizes.

When reflectivity factor $(M_j = M_6)$ and another moment (M_i) are used for the normalization the standard deviation of the fractional error of Fig. 3b is obtained. Again not surprisingly, there are two minima (zero) in the error. As the order *i* is lower, the error at lower (higher) moments decreases (increases). When the order of two moments is far from each other, the overall error is much lower and the error between two moments slightly increases. However, *R* (*n*=3.67) is estimated always with a precision better than 10%. Since reflectivity factor is directly measured from radar, in the application to radar remote sensing we prefer to fix the one moment as reflectivity factor.

b. Connection between scaling normalizations and physical processes.

We now compare the single moment normalization with the double-moment normalization on the data stratified according to precipitation types (stratiform and convective rain). This comparison provides an idea of the feasibility of both normalizations to identify different precipitation types.

Fig. 4a shows the weighted total least square regression (WTLS: Amemiya 1997) of *R* and *Z* for the two types of precipitation. Although the points are weakly separated, the difference in the two regressions is statistically significant. Note the significantly different exponent. Fig. 4b shows the exponent $\gamma(n)$ of the power-law relationship [$M_n = C_{1,n} R^{\gamma(n)}$] between R and all other moments of the indicated order. Again, the two regression lines are clearly distinctive for the convective and stratiform rain. The slope of these two regression lines defines the scaling exponent β of single-moment normalization for the two populations.

The average normalized DSDs $\overline{h(x_2)}$ in Fig. 5a are remarkably stable, indicating that in this case stratiform and convective rain do not generate distinctive shapes of DSDs. $\overline{h(x_2)}$ is slightly different from the normalized exponential DSDs. These DSD shapes are quite consistent with those of TT. The next question is how well two moments or $N_0^{'}$ and $D_m^{'}$ jointly contain information on the scaling exponent β that nicely separates the two rain regimes in the single-moment normalization. TT showed that $N_0^{'}$ and $D_m^{'}$ from the two regimes of tropical rain are well separated so that they are good indicators for the classification. In Fig 5b. , we see the correlation between $N_0^{'}$ and $D_m^{'}$. The separation of the two types of precipitation in the ($N_0^{'}$, $D_m^{'}$) space is poor. Convective rain shows no correlation (determination coefficient r^2 =0.01) and wide distribution with an upper limit of $N_0^{'}$ at $N_0^{'} = 3x10^2$ m^{-3}mm^{-1}. Some points from convective precipitation are mixed with those from stratiform rain.

5. THE FUNCTIONS $g(x_1)$, $h(x_2)$ AND A SCALING MODEL DISTRIBUTION.

From original generalized gamma DSD suggested by Auf der Maur (2001), we obtain the following inherent scaling property and the normalized form of generalized gamma DSDs.

$$M_n = C_{GG,2,n} M_i^{\frac{n-j}{i-j}} M_j^{\frac{i-n}{i-j}} \tag{9}$$

$$N(D) = N_0^{'} h_{GG,(i,j,\mu,c)}(D/D_m^{'})$$

$$h_{GG,(i,j,\mu,c)}(x_2) = c\Gamma_i^{\frac{j+c\mu}{i-j}} \Gamma_j^{\frac{-i-c\mu}{i-j}} x_2^{c\mu-1} \exp\left[-\left\{\frac{\Gamma_i}{\Gamma_j}\right\}^{c/(i-j)} x_2^c\right] \tag{10}$$

$$\Gamma_i = \Gamma(\mu + i/c);\ \Gamma_j = \Gamma(\mu + j/c)$$

where $C_{GG,2,n} = [\Gamma(\mu + i/c)]^{\frac{j-n}{i-j}} [\Gamma(\mu + j/c)]^{\frac{n-i}{i-j}} \Gamma(\mu + n/c)$ and $x_2 = D/D_m^{'}$. Thus, the generalized gamma DSDs also satisfies scaling properties. Since all naturally occurring DSDs can be reasonably well described by the generalized gamma DSD, this suggest a very general description of all types of DSDs within the scaling framework.

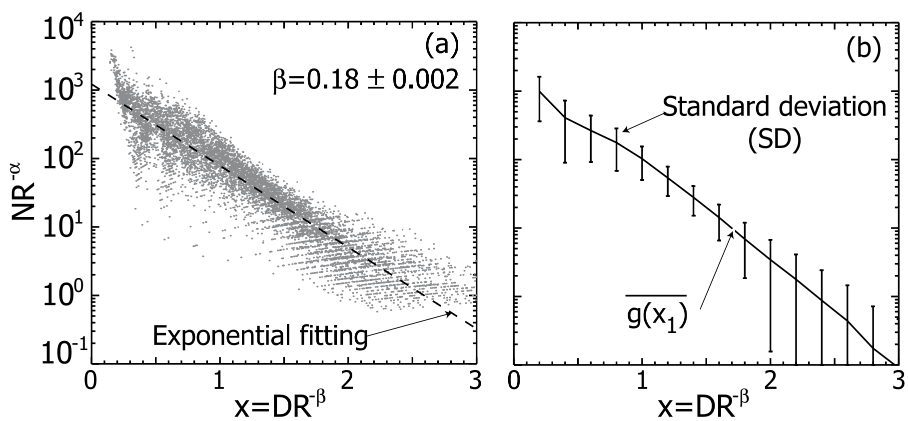

Fig.1: Single-moment normalization on observed DSD data. (a): Scattergram of all normalized DSDs with M_i=R. An exponential adjustment is shown as dashed line. (b): The average $\overline{g(x_1)}$ of the data points in (a) with bars (dark solid line) indicating standard deviation.

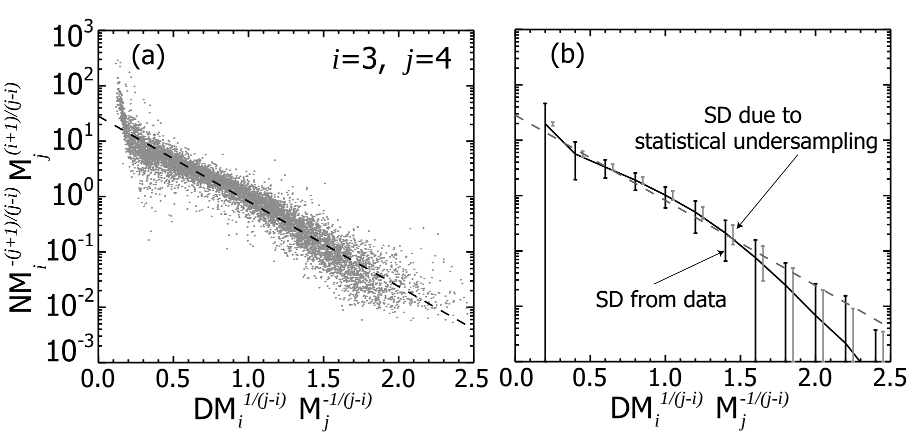

Fig. 2: Two moment normalization on the observed DSD data. (a): Scattergram of all normalized DSDs with M_i=M_3 and M_j=M_4. An exponential adjustment is shown as dashed line. (b): The average $\overline{h(x_2)}$ of the data points in (a) with dark vertical bars indicating standard deviation. The standard deviation due to the statistical fluctuation is shown as the less dark vertical bars.

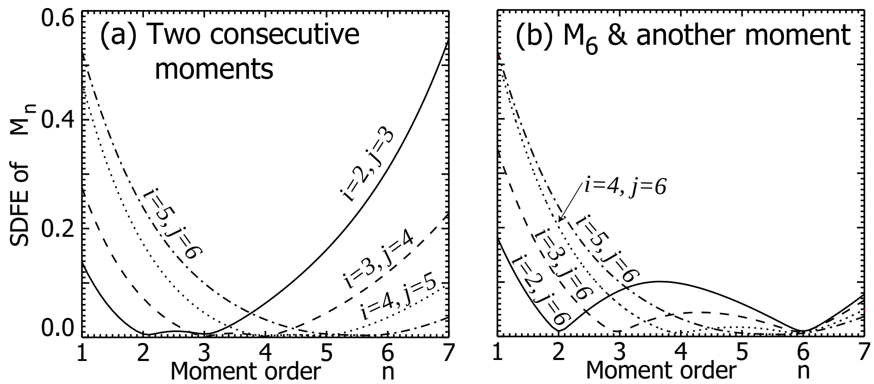

Fig. 3: (a) Standard deviation of the fractional error (SDFE) in the estimate of the n^{th} moment from the average normalized drop size distribution $\overline{h(x_2)}$ when the indicated two consecutive moments are used for the normalization. (b) Same when reflectivity and any other moment are used for the normalization

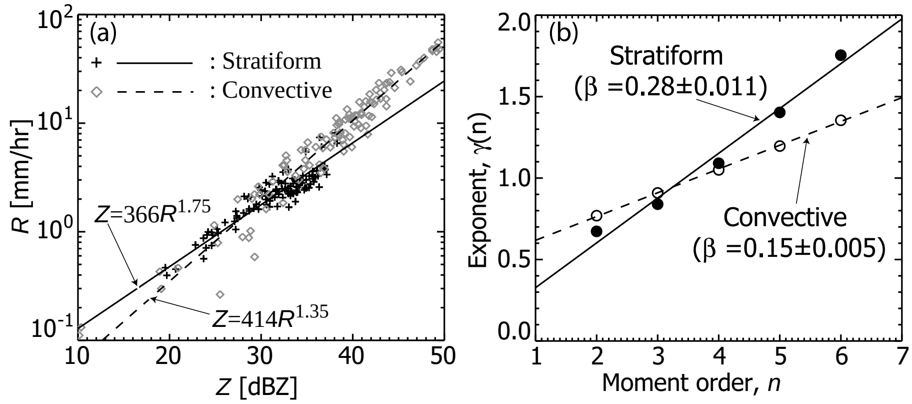

Fig. 4: (a) R-Z WTLS regressions for stratiform and convective rain. (b) Exponent $\gamma(n)$ of $M_n = C_{1,n}R^{\gamma(n)}$ as a function of n. The scaling exponent β is determined by the slope in $\gamma(n)$ vs n [$\gamma(n) = \alpha + (n+1)\beta$].

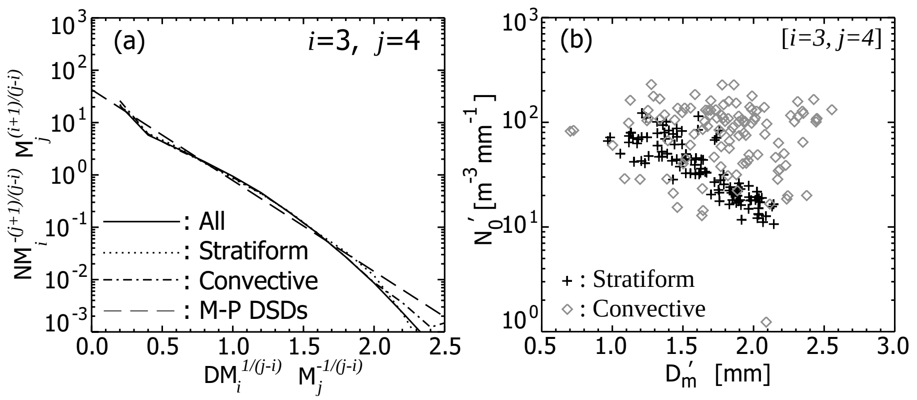

Fig. 5 (a) Comparison of $\overline{h(x_2)}$ from different rain regimes. The normalized exponential DSD is also shown as the long dashed line. Note the remarkably stable shape. (b): Distribution of point in the ($N_0^{'}$, $D_m^{'}$) space.

6. CONCLUSIONS

We have shown here that the ST's and TT's formulations of normalized DSDs are particular cases of the general scaling normalization presented here in some detail. No functional form of DSDs is imposed in the normalization. Therefore, the general scaling normalization can reveal any stable shape of normalized DSDs. The single-moment scaling normalization applied after a stratification of DSDs according to a likely dominance of a given microphysical process shows that the scaling exponent β is a clear indicator of the processes. However, in the double-moment normalization, the separation of $N_0^{'}$ and $D_m^{'}$ that was a good indication of two rain regimes in TT is poor in our data set. The generalized gamma DSDs also have the scaling properties, indicating the derived double-moment scaling DSD formulation is a general way of describing observed DSDs.

7. REFERENCES

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