25.1

THE USE OF A CARTESIAN TERRAIN-INTERSECTING GRID IN A FOURIER-BASED SOLUTION OF THE HELMHOLTZ PROBLEM OF AN IMPLICIT NONHYDROSTATIC FORECAST MODEL

R. James Purser

Science Applications International Corporation
NOAA/NCEP/EMC, Camp Springs, MD

Stephen J. Thomas

National Center for Atmospheric Research
Boulder, CO

1. INTRODUCTION

A fully- or semi-implicit nonhydrostatic model (e.g., Kar et al. 2004) requires that a solution to a Helmholtz equation be obtained at every time step. In the terrain-following coordinates employed by most models (e.g., Yeh et al. 2002) the Helmholtz operator assumes a general non-separable form, even when the model state is arbitrarily close to one of stably stratified rest, and the direct application of efficient Fourier transform methods is precluded. We are examining a solution strategy in which the Helmholtz forcing is first interpolated to a Cartesian terrain-intersecting grid, so that the interior solution operator very closely approximates a separable, symmetric and horizontally homogeneous operator of algebraically simple and constant coefficients, which a Fourier method then efficiently solves. The reconciliation of the original boundary conditions over the terrain surface is then accomplished, to a very close approximation at least, by the construction, via a surface transform of the diagnosed residual from a preliminary trial solution, of a corrective forcing. A second iteration of the Fourier solver will now produce a solution in the Cartesian grid essentially consistent with the terrain boundary condition, so that a further interpolation back to the model's native grid achieves the original objective. By permitting the vertical portion of the grid for the solver to differ from that of the model we are free to choose from a much more general class of model vertical coordinates than would be the case if the choice were to be dictated by the requirement of numerical efficiency of the Helmholtz solver constrained to use this same grid.

In section 2 we summarize the derivation of the Helmholtz problem and its lower boundary condition. Section 3 shows how the homogeneous Helmholtz problem may be solved through the use of double Fourier transformation in the horizontal. Section 4 describes the strategy used to tackle the *non*-homogeneous problem that pertains to the case where the physical lower boundary is not uniformly horizontal. A progress report of this ongoing study will be presented at the conference.

Corresponding author address: R. J. Purser, W/NP2 RM 207, WWBG, 5200 Auth Road, Camp Springs, MD 20746

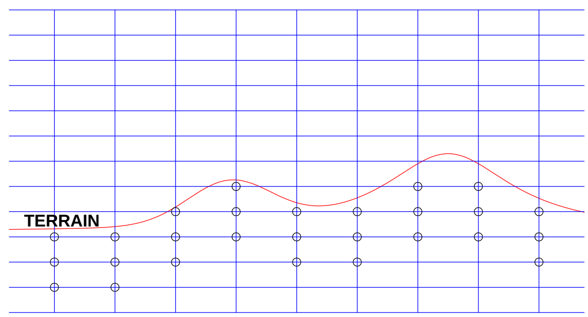

Figure 1. Schematic depiction of the embedding of terrain within the cartesian grid employed for fast-solution of a homogeneous Helmholtz problem. Internal forcing terms applied at the circled locations below the terrain allow the helmholtz solution in the region above the terrain to become approximately consistent with the inhomogeneous problem in which vertical Robin-type conditions are satisfied along the terrain surface.

2. THE HELMHOLTZ EQUATION FOR NONHYDROSTATIC DYNAMICS

In Purser and Kar (2002) it is shown how the non-hydrostatic perturbation equations about a static isothermal atmosphere may be expressed in a rather simple form when the perturbation variables are carefully scaled to make the total energy proportional to the integrated sum of squares of these variables. In three dimensions, with Cartesian coordinates, (x, y, z), with scaled velocity components (u, v, w) scaled pressure perturbation π and scaled potential temperature perturbation θ, the resulting perturbation equations become:

$$\frac{\partial u}{\partial t} = -c\frac{\partial \pi}{\partial x}, \tag{1a}$$

$$\frac{\partial v}{\partial t} = -c\frac{\partial \pi}{\partial y}, \tag{1b}$$

$$\frac{\partial w}{\partial t} = -c\left(\frac{\partial}{\partial z} + \frac{1}{L}\right)\pi + N\theta, \tag{1c}$$

$$\frac{\partial \pi}{\partial t} = -c\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - c\left(\frac{\partial}{\partial z} - \frac{1}{L}\right)w, \tag{1d}$$

$$\frac{\partial \theta}{\partial t} = -Nw. \tag{1e}$$

In these equations the constants, c, L and N are, respectively, the sound speed, the "Lamb height" and the Brunt-Väisälä frequency, all defined in Purser and Kar (2002). The bottom boundary, being an impermeable surface, forces the wind vector there to become tangential to the terrain slope:

$$w|_Z = u\frac{\partial Z}{\partial x} + v\frac{\partial Z}{\partial y}. \tag{2}$$

Suppose the time discretization of a generic prognostic equation,

$$\frac{\partial \psi}{\partial t} = F,$$

takes the form,

$$\frac{\psi_n - \psi_{n-1}}{\delta t} = \frac{(1-\epsilon)}{2}F_{n-1} + \frac{(1+\epsilon)}{2}F_n,$$

or, equivalently,

$$\psi_n - \psi_{n-1} = (\delta t - \tau)F_{n-1} + \tau F_n, \tag{3}$$

where the de-centering parameter ϵ is now absorbed into the time step fraction τ defined by:

$$\tau = \frac{(1+\epsilon)\delta t}{2}. \tag{4}$$

For the perturbation equations (1) this time discretization can be shown to imply that a Helmholtz equation in one of the variables, say π, at the new time level n is obtained with the form (dropping subscript n):

$$A\pi - c^2\tau^2\left(\nabla_h^2 + B\frac{\partial^2 \pi}{\partial z^2}\right) = R, \tag{5}$$

where,

$$A \equiv 1 + \frac{c^2\tau^2}{(1+N^2\tau^2)L^2}, \tag{6a}$$

$$B \equiv \frac{1}{(1+N^2\tau^2)}, \tag{6b}$$

$$\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{7}$$

and R denotes the accumulated right-hand side forcing terms. The connection with the vertical velocity is through:

$$c\left(\frac{\partial}{\partial z} + \frac{1}{L}\right)\pi = -\frac{(1+N^2\tau^2)}{\tau}w. \tag{8}$$

This leads to a mixed ("Robin") bottom boundary condition over terrain whose slope can be neglected, and where therefore w vanishes. We now review how Fourier methods may be exploited to solve this problem.

3. FOURIER SOLUTION OF THE HOMOGENEOUS PROBLEM

It is well known that Fourier methods may be applied to find solutions of a broad range of problems in numerical partial differential equations (for example, Schumann and Sweet 1988, Moorthi and Higgins 1992) provided there is a sufficient degree of spatial homogeneity in the form of the problem. In the case of Helmholtz equation (5) with *homogeneous* bottom boundary conditions, we may replace the horizontal Laplacian operator (7) by:

$$\nabla_h^2 \equiv -(k^2 + \ell^2) = -|\boldsymbol{k}|^2, \tag{9}$$

where $(k, \ell) \equiv \boldsymbol{k}$ are the horizontal wave numbers associated with the double Fourier transform:

$$\tilde{\pi}(\boldsymbol{k}) = \int\int \pi(x,y)\exp[-i(kx+\ell y)]dx\,dy. \tag{10}$$

Then (5) reduces to an ordinary differential equation in z for each Fourier component:

$$\hat{A}\tilde{\pi} - \hat{B}\frac{d^2\tilde{\pi}}{dz^2} = \tilde{R}, \tag{11}$$

where

$$\hat{A} = A + c^2\tau^2(|\boldsymbol{k}|^2), \tag{12a}$$

$$\hat{B} = c^2\tau^2 B. \tag{12b}$$

With suitable boundary conditions, top and bottom, (11) may be solved numerically by a fast tri-diagonal solver for each Fourier component. However, the horizontally homogeneous problem to which this approach applies is not quite the problem we wish to solve. The next section outlines an iterative adaptation of the Fourier method that treats the inhomogeneous problem.

4. SOLVING THE HORIZONTALLY INHOMOGENEOUS PROBLEM

If the numerical grid domain is expanded, as in Fig. 1, to properly contain the physical (above terrain) portion together with a few additional subterranean grid points, then in principle, we are at liberty to include additional right-hand side forcing terms in (5) below terrain in order that: (i) the proper physical bottom boundary condition implied by (2) holds at each vertical grid-column's intersection with the terrain surface; (ii) the numerical grid's bottom boundary condition remains horizontally homogeneous. In this case it is *still* feasible to employ the numerically efficient Fourier solution procedure over the expanded domain.

Since there is precisely one physical bottom boundary condition per vertical grid column, we need one (and only one) discretionary degree of freedom per column in setting up the auxiliary forcing. However, we are free to distribute this forcing in proportion to any weighting profile that fits the vertical space available between the physical and numerical lower boundaries. The circles of Fig. 1 show a possible choice for the locations where these weight functions are nonzero, which happens to be the first three grid points below ground in each column in this example.

In order to maintain, as far as possible, a homogeneous relationship beween the vertical auxiliary weight profile and the terrain altitude in all the columns, we suggest that: (i) the weight profile be essentially a vertical interpolation stencil to a target point located a constant distance below the terrain altitude, $Z(x,y)$; (ii) the numerical grid's bottom boundary condition for (5), and hence for (11) be the "natural" condition that simulates an infinite downward continuation of the domain with a vanishing right-hand side forcing. In the case of (11) this latter condition, applied at the bottom boundary $z = z^*$ of the numerical domain, can be seen to be:

$$\left.\frac{d\tilde{\pi}}{dz}\right|_{z=z^*} = (\hat{A}/\hat{B})^{1/2}\tilde{\pi}, \tag{13}$$

as a consequence of the factoring of the left-hand side of (11) into a symmetric pair of first-order operators associated with exponentially decaying characteristic solutions in either the upward or downward directions:

$$\left(\sqrt{\hat{A}} + \sqrt{\hat{B}}\frac{d}{dz}\right)\left(\sqrt{\hat{A}} - \sqrt{\hat{B}}\frac{d}{dz}\right)\tilde{\pi} = \tilde{R}. \tag{11$'$}$$

The proposed solution procedure is summarized as follows:

i) Interpolate the incremental implicit equations for each of the prognostic variables from model variables and grid to rescaled perturbation variables and expanded Cartesian numerical grid;

ii) Form an interim solution by the Fourier method without the auxiliary forcing;

iii) Diagnose the residual error in the physical boundary condition (2) at the interpolated altitude $Z(x,y)$ corresponding to the terrain surface;

iv) Form the auxiliary subterranean forcing terms in response to the diagnostics from step (iii) and solve by the Fourier method a second time.

v) Back-substitute adjustments to other scaled perturbation variables and interpolate them back to the model's grid and variables.

In step (iv) the auxiliary forcing is formed as the product of each column's weight profile (see discussion above) and a modulating function, $s(x,y)$, of horizontal position. If the diagnostic of residual error in step (iii) is denoted, $e(x,y)$, then, passing from step (iii) (diagnosing error) to step (iv) (correcting that error) requires that we are able to carry out the linear inversion involving the Jacobian, or sensitivity, of each component of e with respect to each component of s. Fortunately, for terrain that is not too steep, a good approximation to this inversion can be obtained by working in the horizontal Fourier domain, where the corresponding Jacobian of $\tilde{e}(\boldsymbol{k}_e)$ with respect to $\tilde{s}(\boldsymbol{k}_s)$ becomes approximately diagonal:

$$\frac{\partial \tilde{e}(\boldsymbol{k}_e)}{\partial \tilde{s}(\boldsymbol{k}_s)} \approx 0,\ \boldsymbol{k}_e \neq \boldsymbol{k}_s, \tag{14}$$

and the inversion is therefore relatively trivial.

5. DISCUSSION AND CONCLUSION

The distinction we have introduced between the grid used by the model's dynamics and the more regular Cartesian grid used purely in the context of the Helmholtz problem means that adopting this proposed solution strategy frees us from the constraint that the model grid is also one in which a numerical elliptic solver is easy to apply. In particular, the present strategy may be particularly well-suited to a model whose vertical grid coordinates has, at least in part, the features of an isentropic framework, where a large variation can potentially exist in the thickness of model layers from one location to another. The advantages of such a framework for modeling have been discussed, for example in Uccellini et al. (1979); Bleck and Benjamin (1993); Johnson et al. (1993).

We are presently at an early stage in the investigation of this strategy, so that, at the time of writing, it remains unclear whether the procedure suggested here is competitive, in terms of numerical efficiency, with existing elliptic solvers (e.g., Thomas et al. 2003). Also, it is not yet clear whether the approach is seriously jeopardized in practice by steep topography, by an over-simplification of basic linearization, or other difficulties. We expect to be able to say more about this at the conference, in the light of our ongoing experience with the approach described.

6. ACKNOWLEDGMENTS

This work was partially supported by the NSF/NOAA Joint Grants Program of the US Weather Research Program. This research is also in response to requirements and funding by the Federal Aviation Administration (FAA). The views expressed are those of the authors and do not necessarily represent the official policy or position of the FAA.

REFERENCES

Bleck, R., and S. Benjamin, 1993: Regional weather prediction with a model combining terrain-following and isentropic coordinates. Part I. *Mon. Wea, Rev.,* **121,** 1770–1785.

Johnson, D. J., T. H. Zapotocny, F. M. Reames, B. J. Wolf, and R. B. Pierce, 1993: A comparison of simulated precipitation by hybrid isentropic-sigma and sigma models. *Mon. Wea. Rev.,* **121,** 2088–2114.

Kar, S. K., R. J. Purser, S. Gopalakrishnan, and G. DiMego, 2004: *(Preprint) AMS 20th Conference on Weather Analysis and Forecasting and 16th Conference on Numerical Weather Prediction,* 12th–15th January 2004, Seattle, WA. 4 pp.

Moorthi, S., and R. W. Higgins, 1992: A fast direct solver for a class of two-dimensional separable elliptic equations on the sphere. *NASA Technical Memorandum* 104567. NASA Goddard, Greenbelt, MD 20771. 30 pp.

Purser, R. J., and S. K. Kar, 2002: Radiative upper-boundary condition for a non-hydrostatic atmosphere. *Quart. J. Roy. Meteor. Soc.,* **128,** 1343–1366.

Schumann, U., and R. Sweet, 1988: Fast Fourier transforms for direct solution of Poisson's equation with staggered boundary conditions. J. Comput. Phys., **75,** 123–137.

Thomas, S. J., J. P. Hacker, P. Smolarkiewicz, and R. B. Stull, 2003: Spectral preconditioners for nonhydrostatic atmospheric models. *Mon. Wea. Rev.*, **131,** 2464–2478.

Uccellini, L. W., D. R. Johnson, and R. E. Schlesinger, 1979: An isentropic and sigma coordinate hybrid numerical model: Model development and some initial tests. *J. Atmos. Sci.,* **36,** 390–414.

Yeh, K.-S., J. Côté, S. Gravel, A. Méthot, A. Patoine, M. Roch, and A. Staniforth, 2002: The CMC-MRB Global Environmental Multiscale (GEM) model. Part III: Nonhydrostatic formulation. *Mon. Wea. Rev.,* **130,** 339–356.