25.2 A FULLY-IMPLICIT SEMI-LAGRANGIAN NONHYDROSTATIC MODEL EMPLOYING A PRESSURE-HYBRID VERTICAL COORDINATE

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1. INTRODUCTION

We present the formulation of a fully implicit, semi-Lagrangian, 3D nonhydrostatic, limited area, finite difference grid point model in a terrain following, hydrostatic pressure based hybrid vertical coordinate. The model is being developed as a contribution to the Weather Research and Forecasting (WRF) model development initiative at NCEP/EMC.

A semi-implicit semi-Lagrangian (SISL) scheme provides a computationally efficient method for solving the dynamical equations of the atmosphere, both in limited-area NWP and global GCM-type models. In a SISL scheme, the so-called *nonlinear* terms (essentially the terms that are non-advective and left over from the semi-implicit linearization of the governing equations) are treated in a time-explicit manner. In a fully-implicit semi-Lagrangian (FISL) scheme, such nonlinear terms are treated in an implicit manner so that the FISL scheme becomes 'more' stable compared to a SISL scheme.

Since the introduction of hydrostatic pressure (Laprise 1992) as an independent vertical coordinate for use in nonhydrostatic models, a number of *Eulerian*, nonhydrostatic, limited-area NWP models that employ a terrain-following, hydrostatic-pressure based vertical coordinate, have been developed (e.g., Bubnova et al. 1995, Skamarock et al. 2001, Janjic et al. 2001).

Recently, a fully-implicit, semi-Lagrangian, nonhydrostatic, global grid-point model has been developed by Yeh et al. (2002), that employs a terrain-following, reference-state hydrostatic-pressure based vertical coordinate. We are currently developing a FISL nonhydrostatic, limited-area, grid-point model at NCEP/EMC. In our model, we have essentially adapted the FISL scheme proposed by Yeh et al. (2002), to our continuous set of model equations. However, unlike their model, we employ (a) a terrain-following, hydrostatic-pressure based hybrid vertical coordinate, (b) a conformal map-coordinate in the horizontal, (c) 3D Coriolis force terms in the momentum equations, (d) a 3D unstaggered grid in space, and (e) an independently developed space-discretization of the model equations.

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2. MODEL FORMULATION

2.1 *Continuous equations*

The governing equations for a fully-compressible, nonhydrostatic, moist-diabtic atmosphere can be written as

Momentum equations

$$d_t u = F_u \equiv f_z v - \gamma f_y w - m[RT_v \partial_x \ln p + (\partial_\eta p / \partial_\eta \tilde{p}) \partial_x \phi] + F_x , \quad (1)$$

$$d_t v = F_v \equiv -f_z u + \gamma f_x w - m[RT_v \partial_y \ln p + (\partial_\eta p / \partial_\eta \tilde{p}) \partial_y \phi] + F_y , \quad (2)$$

$$\gamma d_t w = F_w \equiv \gamma(f_y u - f_x v) - g(1 - \partial_\eta p / \partial_\eta \tilde{p}) + \gamma F_z , \quad (3)$$

Thermodynamic energy equation

$$d_t \ln T_v - \kappa d_t \ln p = Q/(c_p T_v) , \quad (4)$$

Definition of w or ϕ*-tendency equation*

$$d_t \phi = gw , \quad (5)$$

Mass continuity equation

$$d_t(\partial_\eta \tilde{p}) + (D + \partial_\eta \dot{\eta}) \partial_\eta \tilde{p} = 0 , \quad (6)$$

Definition of hydrostatic pressure $\tilde{p}$

$$\partial_\eta \phi = -(RT_v / p) \partial_\eta \tilde{p} , \quad (7)$$

Moisture continuity equation

$$d_t q = -Q_2 / L . \quad (8)$$

Here (x, y) denote the Cartesian map-projection horizontal coordinate, with a map factor m; z denotes the height above the mean sea level and η denotes the vertical coordinate, defined later, based on the hydrostatic pressure, $\tilde{p}$; u, v, and w denote the wind components in the

x-, y-, and η-directions, respectively; T_v, ϕ, q, and p respectively denote the virtual temperature, the geopotential (gz), the specific humidity, and the total pressure. In the above equations, the partial derivatives wrt x, y, and t (time) are taken over constant η surfaces. In the momentum equations (1)-(3), the components of the 3D Coriolis force are combined with the metric terms and the coefficients of the resulting terms are denoted by

$$f_x \equiv e\sin\alpha - v/a \,,$$

$$f_y \equiv e\cos\alpha + u/a \,,$$

$$f_z \equiv f + u\partial_y m - v\partial_x m \,,$$

where

$e \equiv 2\Omega\cos\varphi$, $f \equiv 2\Omega\sin\varphi$, and $\tan\alpha \equiv -(\partial_y\lambda/\partial_y\varphi)\cos\varphi$. Here λ and φ denote the longitude and latitude, respectively. In the mass continuity equation (6), D denotes horizontal divergence defined by

$$D \equiv m^2[\partial_x(u/m) + \partial_y(v/m)] \,.$$

For an arbitrary variable, ψ, the operator $d_t()$ is defined by

$$d_t\psi \equiv \dot{\psi} = [\partial_t + m(u\partial_x + v\partial_y) + \dot{\eta}\partial_\eta]\psi \,,$$

where $\dot{\eta} \equiv d_t\eta$ denotes the η-coordinate vertical velocity. The F terms in (1)-(3) denote friction; Q in (4) denotes the heating rate; Q_2 and L in (8) denote the apparent moisture sink and latent heat of condensation, respectively. The virtual temperature T_v in (4) is defined by $T_v \equiv T[1 + (R_v/R - 1)q]$. The physical constants a, g, Ω, R, c_p, $\kappa \equiv R/c_p$, and R_v have their usual meaning.

The constant parameter γ in (1)-(3) is set to unity for a nonhydrostatic model and it is set to zero for a hydrostatic model. For a hydrostatic model, $p \equiv \tilde{p}$, so that (3) is trivially satisfied and (5) becomes decoupled (and thus excluded) from the governing equations and ϕ is then diagnosed using (7), that reduces to the *hydrostatic* equation $\partial_\eta\phi = -RT_v\partial_\eta\ln\tilde{p}$.

The vertical coordinate (η) is defined by

$$\tilde{p}(x, y, \eta, t) = A(\eta)p_T + B(\eta)\tilde{p}_*(x, y, t) \,, \tag{9}$$

where p_T and $\tilde{p}_*$, respectively, denote a constant pressure at the model top and the hydrostatic pressure at the earth's surface (model bottom). Without loss of generality, we assume that η varies from zero at the surface to unity at the model top. The functions A and B are arbitrary, but satisfy the restrictions

$$A(0) = 0 \,;\; B(0) = 1 \,;\; A(1) = 1 \,;\; B(1) = 0 \,. \tag{10}$$

When A and B are linear functions of η, defined by

$$A(\eta) = \eta \,;\; B(\eta) = 1 - \eta \,, \tag{11a}$$

equation (9) yields

$$\eta = (\tilde{p}_* - \tilde{p})/(\tilde{p}_* - p_T) \,. \tag{11b}$$

Equation (11b) is identified as a modified form of Phillip's (1957) σ-coordinate, where the model top is at a nonzero constant pressure. This particular form of η has been used in a number of *Eulerian* nonhydrostatic limited-area grid point models, including the NCAR WRF mass-coordinate model (Skamarock et al. 2001) and the NCEP Nonhydrostatic meso-scale model (NMM, Janjic et al. 2001).

The earth's surface and model top are assumed to be material surfaces that are also η-coordinates surfaces, so that the lower and upper boundary conditions are

$$\dot{\eta} = 0 \text{ at } \eta = 0 \text{ and } \eta = 1 \,. \tag{12}$$

2.2 *Fully-implicit semi-Lagrangian scheme*

Let us consider a generic prognostic equation

$$d_t\psi = F[\psi(x, y, \eta, t)] \,. \tag{13}$$

Then, along a *backward* 3D trajectory, a two time-level, fully-implicit semi-Lagrangian (FISL) scheme for (13) can be written as

$$\frac{\psi^n - \psi_d^{n-1}}{\Delta t} = \frac{1}{2}[(1+\varepsilon)F^n + (1-\varepsilon)F_d^{n-1}] \,. \tag{14}$$

Here, n and n-1 denote the two time levels, Δt is the time step, and $\varepsilon \in [0, 1]$ denotes the uncentering parameter. The variables with a subscript d are evaluated at the departure point, and the variables without a subscript are carried/evaluated at the grid points.

Introducing the variables

$$\tau \equiv \frac{1+\varepsilon}{2}\Delta t \,;\; \Re \equiv \frac{\psi}{\tau} + \frac{1-\varepsilon}{1+\varepsilon}F \,, \tag{15}$$

we can rewrite (14) as

$$\frac{\psi^n}{\tau} - F^n = \frac{\psi_d^{n-1}}{\tau} + \frac{1-\varepsilon}{1+\varepsilon}F_d^{n-1} \equiv \Re_d^{n-1} \,. \tag{16}$$

Let us now *linearize* F^n as $F^n = L^n + N^n$, where L and N, respectively, denote the *linear* and *nonlinear* parts of F; and then rewrite (16) as

$$\psi^n/\tau - L^n = \Re_d^{n-1} + N^n \equiv S \,. \tag{17}$$

For a traditional *semi-implicit semi-Lagrangian scheme*, N^n is evaluated in a time-explicit manner, using the time extrapolation

$$N^n = 2N^{n-1} - N^{n-2} \,. \tag{18}$$

On the other hand, for the *fully-implicit semi-Lagrangian scheme*, (17) is solved as an iterative problem

$$[\psi^n/\tau - L^n]^{(i)} = \Re_d^{n-1} + [N^n]^{(i-1)}, \tag{19}$$

where the superscript (i) denotes an iterative index.

2.3 *FISL discretization of continuous equations*

Towards this objective, the governing equations (1)-(7) are linearized wrt a resting, hydrostatic, isothermal (constant temperature, T_{00}) reference atmosphere, without surface topography. The geopotential and the hydrostatic pressure of the reference state are given by

$$\phi_0(\eta) = -RT_{00}\ln(\tilde{p}_0/p_{00});$$

$$\tilde{p}_0(\eta) = A(\eta)p_T + B(\eta)p_{00}, \tag{20}$$

where p_{00} is a standard pressure. Then, the dependent variables, ϕ, $\tilde{p}$, T_v, and p are expressed in terms of the reference atmosphere as

$$\phi = \phi_*(x,y) + \phi_0(\eta) + \phi'; \ \tilde{p} = \tilde{p}_0(\eta) + r',$$

$$T_v = T_{00} + T', \ p = \tilde{p}\exp(q'). \tag{21}$$

Here ϕ_* denotes the surface geopotential.

Using (21), the thermodynamic equation (4), the ϕ-tendency equation (5), and the mass continuity equation (6) can be rewritten as

$$d_t[\ln(1+T'/T_{00}) - \kappa\ln\{q' + \ln(1+r'/\tilde{p}_0)\}] = F_T$$
$$\equiv \kappa\dot{\eta}d_\eta\Lambda_0 + Q/(c_pT_v), \tag{22}$$

$$d_t\phi' = F_\phi \equiv RT_{00}\dot{\eta}d_\eta\Lambda_0 + gw$$
$$-m(u\partial_x + v\partial_y)\phi_*, \tag{23}$$

$$d_t(\partial_\eta r') = F_r \equiv -[\partial_\eta(\dot{\eta}d_\eta\tilde{p}_0) + Dd_\eta\tilde{p}_0 +$$
$$(D + \partial_\eta\dot{\eta})\partial_\eta r'], \tag{24}$$

where $\Lambda_0 \equiv \ln\tilde{p}_0$.

Then, applying the scheme (16) to each of the prognostic equations (1), (2), (3), (22), (23), and (24), we obtain

$$u^n/\tau - F_u^n = (\Re_u)_d^{n-1}, \tag{25a}$$

$$v^n/\tau - F_v^n = (\Re_v)_d^{n-1}, \tag{25b}$$

$$\gamma w^n/\tau - F_w^n = (\Re_w)_d^{n-1}, \tag{25c}$$

$$\frac{1}{\tau}[\ln(1+T'^n/T_{00}) - \kappa\{q'^n + \ln(1+r'^n/\tilde{p}_0)\}] - F_T^n$$
$$= (\Re_T)_d^{n-1}, \tag{25d}$$

$$\phi'^n/\tau - F_\phi^n = (\Re_\phi)_d^{n-1}, \tag{25e}$$

$$\partial_\eta r'^n/\tau - F_r^n = (\Re_r)_d^{n-1}, \tag{25f}$$

where $\Re_u \equiv \frac{u}{\tau} + \frac{1-\varepsilon}{1+\varepsilon}F_u$, etc.

The functions F_u^n, F_v^n, F_w^n, F_T^n, F_ϕ^n, and F_r^n appearing above, are then linearized in terms of the following *linear* and *nonlinear* components

$$L_u^n \equiv f_0 v^n - m\partial_x G'^n; \ N_u^n \equiv [F_u - L_u]^n, \tag{26a}$$

$$L_v^n \equiv -f_0 u^n - m\partial_y G'^n; \ N_v^n \equiv [F_v - L_v]^n, \tag{26b}$$

$$L_w^n \equiv g\partial_\eta(\tilde{p}_0 q'^n)/d_\eta\tilde{p}_0; \ N_w^n \equiv [F_w - L_w]^n, \tag{26c}$$

$$L_T^n \equiv \kappa\dot{\eta}^n d_\eta\Lambda_0; \ N_T^n \Leftarrow Q^n/(c_pT_v^n), \tag{26d}$$

$$L_\phi^n \equiv RT_{00}\dot{\eta}^n d_\eta\Lambda_0 + gw^n; \ N_\phi^n \equiv -m(u^n\partial_x + v^n\partial_y)\phi_*, \tag{26e}$$

$$L_r^n \equiv -[\partial_\eta(\dot{\eta}^n d_\eta\tilde{p}_0) + D^n d_\eta\tilde{p}_0];$$
$$N_r^n \equiv -[(D^n + \partial_\eta\dot{\eta}^n)\partial_\eta r'^n], \tag{26f}$$

where

$$G' \equiv \phi' + RT_{00}(q' + r'/\tilde{p}_0), \tag{26g}$$

denotes a perturbation generalized geopotential. In (26a) and (26b), the Coriolis parameter f is linearized as $f = f_0 + f'$, where f_0 is an area-averaged value of f. The symbol $\Leftarrow$ used in (26d) indicates that the form of N_T^n is not final at this stage of the derivation.

Then, substituting (26a) through (26f) in (25a) through (25f), respectively, we obtain

$$u^n/\tau - f_0v^n + m\partial_x G'^n = S_u \equiv (\Re_u)_d^{n-1} + N_u^n, \tag{27a}$$

$$v^n/\tau + f_0u^n + m\partial_y G'^n = S_v \equiv (\Re_v)_d^{n-1} + N_v^n, \tag{27b}$$

$$\gamma w^n/\tau - g\partial_\eta(\tilde{p}_0 q'^n)/d_\eta\tilde{p}_0 = S_w \equiv (\Re_w)_d^{n-1} + N_w^n, \tag{27c}$$

$$\frac{1}{\tau}[\ln(1+T'^n/T_{00}) - \kappa\{q'^n + \ln(1+r'^n/\tilde{p}_0)\}] - \kappa\dot{\eta}^n d_\eta\Lambda_0$$
$$= S_T \Leftarrow (\Re_T)_d^{n-1} + N_T^n, \tag{27d}$$

$$\phi'^n/\tau - (gw^n + RT_{00}\dot{\eta}^n d_\eta\Lambda_0) = S_\phi \equiv (\Re_\phi)_d^{n-1} + N_\phi^n, \tag{27e}$$

$$\partial_\eta r'^n/\tau + [\partial_\eta(\dot{\eta}^n d_\eta\tilde{p}_0) + D^n d_\eta\tilde{p}_0] = S_r$$
$$\equiv (\Re_r)_d^{n-1} + N_r^n. \tag{27f}$$

The two logarithmic terms in (27d) are linearized so that (27d) is further reduced to the from

$$\frac{1}{\tau}[T'^n/T_{00} - \kappa(q'^n + r'^n/\tilde{p}_0)] - \kappa\dot{\eta}^n d_\eta\Lambda_0 = S_T$$
$$\equiv (\Re_T)_d^{n-1} + N_T^n, \tag{28a}$$

where

$$N_T^n \equiv \frac{1}{\tau}[T'^n/T_{00} - \ln(1+T'^n/T_{00})$$
$$-\kappa\{r'^n/\tilde{p}_0 - \ln(1+r'^n/\tilde{p}_0)\}] + Q^n/(c_pT_v^n). \tag{28b}$$

Thus, equations (27a), (27b), (27c), (28a), (27e), and (27f) represent the individual FISL schemes for the prognostic variables $(u, v, w, T', \phi'$, and $\partial_\eta r')^n$, respectively. This set is augmented by a linearized from of the hydrostatic-pressure equation (7), given by

$$\partial_\eta \phi'^n = -RT_{00}[(T'^n/T_{00} - q'^n)d_\eta \Lambda_0 + \partial_\eta(r'^n/\tilde{p}_0)] + \aleph_\phi^n, \quad (29)$$

where the exact form $\aleph_\phi^n$ is not shown. Note that these equations are to be solved in the iterative manner indicated by (19).

Then, the FISL discrete equations (27a), (27b), (27c), (28a), (27e), (27f), (29), and (26g) in terms of the unknown variables $(u, v, w, T', \phi', r', q', \dot{\eta}$, and $G')^n$, are algebraically manipulated so that a single 3D elliptic equation for G'^n is derived:

$$\left[D_h - \frac{\gamma\mu^{-1}(1-\kappa)}{g^2\tau^4\kappa} + \frac{\mu^{-1}}{\kappa R T_{00}\tau^2}\left(D_{\eta\eta} - \frac{1}{4}\right)\right](\tilde{p}_0^{1/2}G'^n) = \hat{Y}_e, \quad (30a)$$

where

$$D_h\psi \equiv \frac{m^2}{1+(f_0\tau)^2}(\partial_{xx} + \partial_{yy})\psi \,;\; \mu \equiv 1 + \frac{\gamma R T_{00}}{\kappa(g\tau)^2},$$

$$D_\eta\psi \equiv \frac{\partial_\eta\psi}{d_\eta\Lambda_0},\; D_{\eta\eta}\psi \equiv D_\eta(D_\eta\psi)\,; \quad (30b)$$

and the exact form of $\hat{Y}_e$ is not shown. We need to solve (30a) for G'^n in conjunction with the back-substitution equations (not shown) for the other unknown variables, in an iterative manner. This in essence completes the FISL formulation as it applies to the continuous equations.

2.4 *On space discretization and other issues*

A 3D unstaggered grid is used in (x, y, η) for spatial discretization; all model variables $(u, v, w, T', \phi', r', q', G', \dot{\eta})$ are carried at each grid point. Second-order centered schemes are used for all non-advective terms and computations on the grid. The 3D backward trajectory calculations employ linear interpolations, whereas 3D quasi-cubic interpolation is used for evaluation of the $\Re_d^{n-1}$ functions.

The 3D elliptic equations (30a) for G'^n is vertically separable, but not so horizontally due to the map factor, $m(x, y)$, appearing in the operator $D_h(\,)$. The 3D elliptic equation (30a) can be vertically decoupled into a set of K (= number of vertical levels in the model) 2D elliptic equations, which can be solved iteratively by a pre-conditioned generalized conjugate residual (GCR) algorithm (e.g., Saad 2003). Alternatively, a pre-conditioned GCR algorithms can be employed to solve the 3D elliptic equation (30a) without going into vertical decoupling.

3. CONCLUSIONS

A fully-implicit, semi-Lagrangian, 3D nonhydrostatic, limited-area, grid-point model in a hydrostatic-pressure based, terrain-following, hybrid vertical coordinate is being developed at NCEP/EMC under the WRF model development initiative. Preliminary results will be presented at the conference.

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