DATA ASSIMILATION BY FIELD ALIGNMENT FOR COHERENT STRUCTURES

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Position errors are ubiquitous in forecasting localized weather phenomena, affecting both objective analysis and verification. classical formulations for data-assimilation will tend to distort the state when adjusted variables do not have a direct impact on position errors, especially in the presence of sparse observations. The sources of position errors are poorly understood and indeed the somewhat crude current practice of bogussing reflects the need to adjust for position errors.

We propose a novel formulation for handling position errors in a preprocessing step to classical data assimilation. In this step, called *field alignment*, the current model state is spatially aligned with observations by adjusting a field of local displacements. This is accomplished by solving an auxiliary variational optimization problem that includes a Tikhonov-type regularization constraint with two *physically valid* weak constraints penalizing non-smooth and divergent displacement-fields and this choice also distinguishes our technique from most methods that seek global constraints.

Further, in contrast to other alignment techniques, our preprocessing step does not explicitly rely on the definition of a feature and displacements are defined in an Eulerian frame and valid in the continuum. The alignment and amplitude adjustment can be interpreted in a Bayesian formalism, albeit whose solutions require several procedural simplifications including a two-step approach to assimilation; alignment followed by amplitude adjustment, and an iterative solution for the non-linear field alignment equation.

a. *Field Alignment*

Start by using auxiliary variables, local displacements, in the observation equation to model the position errors, that is $\mathbf{Y} = H[\mathbf{X}(\mathbf{p}-\mathbf{q})] + \eta$, where $\mathbf{p}$ the vector of position indices and $\mathbf{q}$ a vector representation of a displacement field. We define an quadratic objective (minimum variance, MLE under Gaussian assumption), that slightly modifies a "3D-VAR" objective:

$$J(\mathbf{X}, \mathbf{q}) = L(\mathbf{q}) + \xi^T \mathbf{B}^{-1} \xi + \aleph^T \mathbf{R^{-1}} \aleph$$
$$\xi = \left(\mathbf{X}\left(\mathbf{p}-\mathbf{q}\right) - \mathbf{X^f}\left(\mathbf{p}-\mathbf{q}\right)\right)$$

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$$\aleph = \left(\mathbf{Y} - H\left[\mathbf{X}\left(\mathbf{p}-\mathbf{q}\right)\right]\right) \quad (1)$$

To estimate the true state, we use the Euler-lagrange equations of J, but this produces highly non-linear functions in X and q. A solution is sought by fixing the secondary dependent variable, thereby approximating the full equations with an amplitude only equation and a displacement only equation. We then adopt a sequential approach: align the state with observations then resolve the amplitude. The alignment equation is:

$$\frac{\partial L}{\partial \underline{q}_i} + \nabla \mathbf{X}^{fT}|_{\underline{p}_i - \underline{q}_i} \mathbf{H}^T \mathbf{R^{-1}} \left(H\left[X^f\left(\mathbf{p}-\mathbf{q}\right)\right] - Y\right)|_i = 0 \quad (2)$$

figure

Where $\mathbf{H}$ is the linearized observation operator and $\underline{q}_i \in \Re^n$ is the $n-$dimensional displacement associated with the i^{th} position index $\underline{p}_i$. The optimal estimate $\hat{\mathbf{q}}$ of $\mathbf{q}$ is then used in the "amplitude" equation:

$$\begin{aligned} &\mathbf{B^{-1}}\left(\mathbf{X}(\mathbf{p}-\hat{\mathbf{q}}) - \mathbf{X}^f(\mathbf{p}-\hat{\mathbf{q}})\right) \\ &+\mathbf{H}^T\mathbf{R^{-1}}\left(H\left[\mathbf{X}\left(\mathbf{p}-\hat{\mathbf{q}}\right)\right] - \mathbf{Y}\right) \\ &= 0 \end{aligned} \quad (3)$$

Given a fixed displacement estimate $\hat{\mathbf{q}}$, equation 3 is the Jacobian of a 3D-VAR objective and so the amplitude recovery is simply 3D-VAR. The choice of the constraint L is critical to make this idea work. We posit that position errors can be recovered as smooth flow fields and view them as arising from systematic and large-scale errors, particularly relevant in *background flow* errors. Smoothness suggests a Tikhonov type formulation and, in particular, $L(q)$ is designed with *local constraints*; a gradient penalty term and a divergence penalty term. That is,

$$L(\mathbf{q}) = \frac{w_1}{2}\sum_{i\in\Omega} \mathbf{tr}\{[\nabla \underline{q}_i][\nabla \underline{q}_i]^T\} + \frac{w_2}{2}\sum_{i\in\Omega}[\nabla \cdot \underline{q}_i]^2 \quad (4)$$

These are kinematic constraints, but ones that admit "fluid-like" motions because they prescribe diffeomorphic flows up to the first order[1]. With proper weighting, these constraints handle rotations, translations, shears, skews and stretches. The displacement equation, with first order constraints, is as follows:

[1] Higher order diffeomorphisms can just as easily be configured.

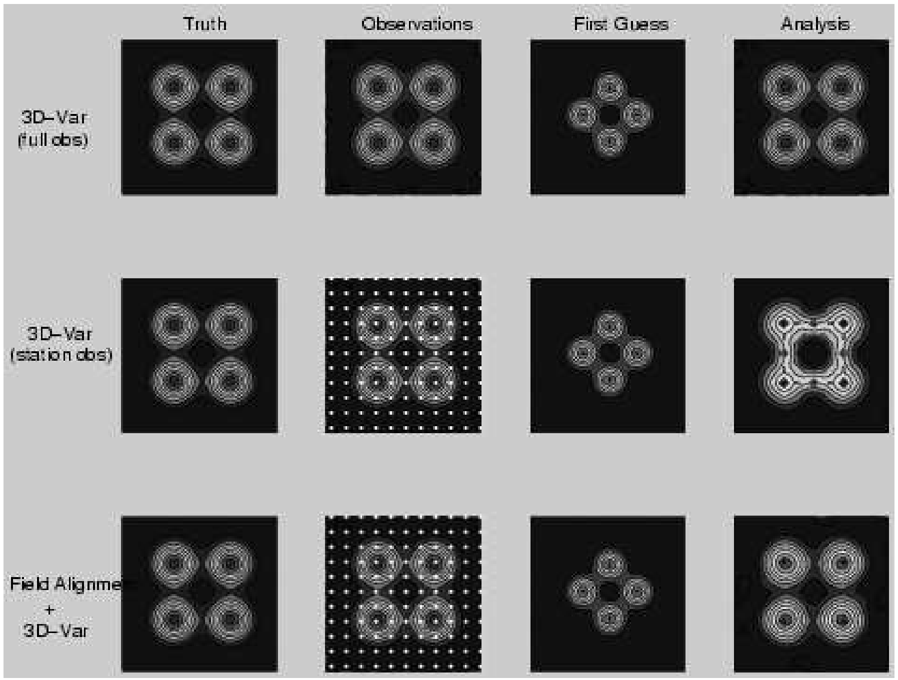

Figure 1: Analysis of a 2D field is shown with full and sparse observations, using 3D-VAR and the field alignment algorithm combined with 3D-VAR. Under sparse observations, automatically aligning the fields first substantially improves analysis.

$$
\begin{aligned}
&w_1\nabla^2 q_i + w_2\nabla(\nabla\cdot q_i) + \\
\nabla X^f|^T_{p_i-q_i}\mathbf{H}^T R^{-1}&\left(H\left[X^f(\mathbf{p}-\mathbf{q})\right]-Y\right)|_i \\
&= 0 \qquad (5)
\end{aligned}
$$

Phenomenologically, Equation 5 introduces a forcing based on the residual between the model-field and observation-field, modulated by the local brightness gradient. The constraints on the displacement field allow the forcing to propagate to a consistent solution. This is an expressly Eulerian approach, individual features are neither identified nor required for matching, although *featuredness or texture* clearly influences the solution. The deformation is defined on the continuum and evolves over iterations.

Equation 5 is also non-linear, and is solved iteratively. During each iteration the forcing term is held constant from the previous iteration and the resulting Poisson equation is solved. The estimate of displacement at each iteration is then used to deform the model-field and the process is repeated again till a small residual is obtained or an iteration limit is reached. The boundary condition for this system is $q_i = 0$ and, in certain atmospheric flows such as the BVE (used here), periodic. The final field is $\mathbf{X}^f(\mathbf{p}-\hat{\mathbf{q}})$ and used as the initial guess for Equation 3.

In Figure 1 the performance of the two-step method is shown. This figure consists of four columns. The left column is the contour plot of truth, showing four vortices. The second column depicts observations, the third is the first guess and the fourth the analysis. The first guess is scaled (in position of vortex centers) by a factor of 1.5 from the truth, is rotated by 45^o, with a mean multiplicative amplitude error of 1.45 at the vortex centers. The observations were generated from truth by introducing 1% uncorrelated noise in amplitude. The background error-covariance was substantially more uncertain than the observational uncertainty.

The first row of Figure 1 demonstrates the performance of 3D-VAR with observations made at every grid point. In this case, because the observational uncertainty is so much smaller, the analysis is perfect. The first guess snaps right to the observations. In the second row, the performance of 3D-VAR with station observations (white dots) is depicted. In particular, the analysis has a little resemblance to truth, but demonstrates substantial smearing effect. To be sure, the effect seen here is not to be interpreted as an "averaging" of two nearly equally uncertain sources, but the inability of the background error covariance to spread information from a sparse observation operator successfully. This has been argued to be the major source of distortion of analysis under position error. The third row depicts the performance of the two-step algorithm developed here. The first guess is aligned and then 3D-VAR is applied. The analysis produced is much more realistic.

We believe, the proposed variational methodology can significantly help practitioners produce smooth physically valid alignments as the first guess in analysis, in a way that accounts for hard to diagnose model errors.