P1.1 LINEAR PREDICTION OF GRAVITY WAVE DRAG IN THE CASE OF GENERIC WIND PROFILES

Miguel A. C. Teixeira*, Pedro M. A. Miranda and Maria Antónia Valente
University of Lisbon, Lisbon, Portugal

1. INTRODUCTION

In order to develop physically sound parameterizations of orographic gravity wave drag for large-scale numerical models, it is useful to study the variation of this force, exerted by mountains on a stratified atmosphere, with the various parameters of the flow. Among these, shear is one of the most important. Although the drag has been calculated analytically for particularly simple non-uniform wind profiles (Smith 1986, Grubišić and Smolarkiewicz 1997), and numerically for more complex wind profiles (Bacmeister and Pierrehumbert 1988, Valente 2000), a general framework to understand which characteristics of the wind variation with height affect the drag, and in what way, has been lacking.

This study uses the linearized, hydrostatic equations of motion to derive closed-form analytical expressions for the drag induced by an axisymmetric mountain and a 2D ridge, for generic wind profiles. This analytical approach enables one to understand more clearly the individual physical processes affecting the drag, through the functional dependence on the various parameters.

2. THEORETICAL MODEL

The proposed model is based on the Taylor-Goldstein equation,

$$\hat{w}'' + \left(\frac{N^2 k_{12}^2}{(Uk_1 + Vk_2)^2} - \frac{U'' k_1 + V'' k_2}{Uk_1 + Vk_2} \right) \hat{w} = 0 \ , \qquad (1)$$

where $\hat{w}$ is the Fourier transform of the vertical velocity perturbation, the primes denote differentiation with respect to z, N is the Brunt-Väisälä frequency of the background flow (assumed constant), $(U(z),V(z))$ is the background wind, (k_1,k_2) is the horizontal wavenumber vector of the internal gravity waves generated by the mountain and $k_{12}=(k_1^2+k_2^2)^{1/2}$. This equation is solved here using the WKB approximation, which formally assumes that the wind velocity varies over a scale that is much larger than the vertical wavelength of the waves. However, in practice, it is shown that the model is accurate even when this condition is not satisfied, or is only marginally satisfied.

In order for the wind variation with height to have any impact on the drag, it is necessary that the WKB solution be extended to second-order in the small perturbation parameter ε (Teixeira et al. 2004):

$$\hat{w} = \hat{w}(z=0) e^{i \int_0^z \left[m_0(\varepsilon z') + \varepsilon m_1(\varepsilon z') + \varepsilon^2 m_2(\varepsilon z') \right] dz'} \ . \qquad (2)$$

That is why previous analytical treatments using the first-order WKB solution (which is the most widely known) have failed to capture this effect (Grisogono 1994, Shutts 1995). m_0, m_1, m_2 are the zeroth-, first- and second-order coefficients of the series expansion of the vertical wavenumber of the gravity waves in powers of ε.

Equation (1) is subject to the boundary condition

$$\hat{w}(z=0) = i(U_0 k_1 + V_0 k_2)\hat{\eta} \ , \qquad (3)$$

where (U_0, V_0) is the wind at the surface and $\hat{\eta}$ is the Fourier transform of the terrain elevation. The radiation boundary condition at $z \to +\infty$ completes the specification of this problem.

Once the solution to (1) is determined, the Fourier transform of the pressure perturbation associated with the waves can be obtained by

$$\hat{p} = i \frac{\rho_0}{k_{12}^2} \left[(U' k_1 + V' k_2)\hat{w} - (Uk_1 + Vk_2)\hat{w}' \right], \qquad (4)$$

where ρ_0 is a reference density.

The gravity wave drag exerted by the atmosphere on the mountain (which has the same magnitude and the opposite sign of the drag exerted by the mountain on the atmosphere) is calculated using

$$(D_x, D_y) = 4\pi^2 i \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (k_1, k_2)\hat{p}^*(z=0)\hat{\eta}\, dk_1 dk_2 \qquad (5)$$

for an isolated mountain. On the other hand, the drag per unit length for a 2D ridge aligned in the y direction is

$$D_x = 2\pi i \int_{-\infty}^{+\infty} k_1 \hat{p}^*(z=0)\hat{\eta}\, dk_1 \ , \qquad (6)$$

where the asterisk denotes complex conjugate.

Subject to the WKB assumption, the drag is found to depend on the first and second vertical derivatives of the wind velocity at the surface, being given by

$$D_x = D_{0x} \left[1 - \frac{1}{32} \left(3\frac{U_0'^2}{N^2} + \frac{V_0'^2}{N^2} + 2\frac{V_0}{U_0}\frac{U_0' V_0'}{N^2} \right) - \frac{1}{16} \left(3\frac{U_0 U_0''}{N^2} + \frac{V_0}{U_0}\frac{V_0 U_0''}{N^2} + 2\frac{V_0 V_0''}{N^2} \right) \right], \qquad (7)$$

$$D_y = D_{0y} \left[1 - \frac{1}{32} \left(3\frac{V_0'^2}{N^2} + \frac{U_0'^2}{N^2} + 2\frac{U_0}{V_0}\frac{U_0' V_0'}{N^2} \right) - \frac{1}{16} \left(3\frac{V_0 V_0''}{N^2} + \frac{U_0}{V_0}\frac{U_0 V_0''}{N^2} + 2\frac{U_0 U_0''}{N^2} \right) \right], \qquad (8)$$

for an axisymmetric mountain and by

$$D_x = D_{0x} \left(1 - \frac{1}{8}\frac{U_0'^2}{N^2} - \frac{1}{4}\frac{U_0 U_0''}{N^2} \right), \qquad (9)$$

* *Corresponding author address*: Miguel A. C. Teixeira, Centro de Geofísica da Universidade de Lisboa, Edifício C8, Campo Grande, 1749-016 Lisbon, Portugal

$$D_y = 0\,, \tag{10}$$

for a 2D ridge, where (U_0', V_0') and (U_0'', V_0'') are the first and second derivatives of the background wind at the surface. (D_{0x}, D_{0y}) denotes the gravity wave drag in the absence of any shear (i.e. for a constant wind).

3. RESULTS

As shown by (7)-(10), the drag normalized by its value in the absence of shear only depends on the characteristics of the flow at the surface, and is independent of the detailed shape of the orography, as long as this is axisymmetric or 2D. This is an important property, which is only valid for hydrostatic flow.

In order to test these drag expressions, 3 different flows are considered next, all of them with constant Richardson number (*Ri*). In all the cases addressed, (7)-(10) simplify considerably, so that the correction to the drag due to the vertical wind variation is expressed in terms of *Ri*. The resulting expressions are compared next with results from simulations of a mesoscale non-hydrostatic numerical model (NH3D - see Miranda and James 1992), for approximately linear and hydrostatic conditions.

3.1. *Linearly decreasing wind – axisymmetric mountain*

Consider a wind profile of the form

$$U = U_0 - \alpha z\,,$$
$$V = U_0\,, \tag{11}$$

where α and U_0 are constants. In this case, (7)-(8) reduce to

$$D_x = D_{0x}\left(1 - \frac{3}{32Ri}\right),$$
$$D_y = D_{0y}\left(1 - \frac{1}{32Ri}\right), \tag{12}$$

where $Ri=N^2/\alpha^2$. The two drag components therefore decrease as *Ri* becomes smaller, but the *x* component decreases faster.

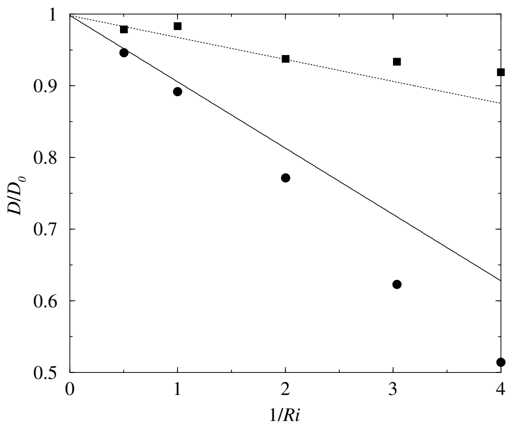


Figure 1. Normalized drag as a function of Ri^{-1} for the flow (11). Lines: eq. (12), symbols: NH3D.

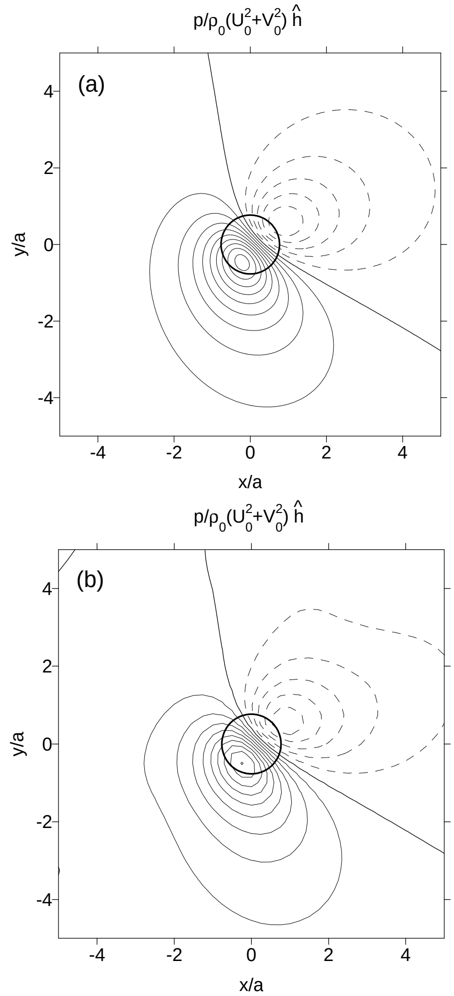


Figure 2. Normalized pressure perturbation at *Ri*=0.5 for the flow (11). Solid lines: positive values, dashed lines: negative values. (a) present model, (b) NH3D.

The drag is also not aligned with the surface wind. Fig. 1 shows a comparison between the drag given by (12) and by the NH3D numerical model. The agreement is quite good, except for very small *Ri*, as would be expected in a WKB model.

Cross-sections of the pressure perturbation at the surface for a bell-shaped mountain are shown in fig. 2 (*a* is the width of the mountain and $\hat{h}$ is its dimensionless height). Relative to the pressure perturbation in a flow with constant wind (Smith 1980), the pressure dipole is weaker, more asymmetric and with the maximum displaced towards the mountain peak (cf. Grubišić and Smolarkiewicz 1997). These features explain the lower drag in fig. 1, and are responsible for its misalignment with the surface wind. In fig. 2, the agreement of the present analytical model with the NH3D model is remarkable.

3.2 *Wind that turns with height – axisymmetric mountain*

For a wind profile of the form

$$U = U_0 \cos(\beta z),$$
$$V = U_0 \sin(\beta z), \tag{13}$$

where β is a constant, (7)-(8) reduce to

$$D_x = D_{0x}\left(1 + \frac{5}{32Ri}\right),$$
$$D_y = 0, \tag{14}$$

where $Ri = N^2/(U_0\beta)^2$. So, although the wind changes direction with height, the drag is predicted to have the direction of the surface wind (unlike the previous case). More importantly, the drag increases as *Ri* decreases.

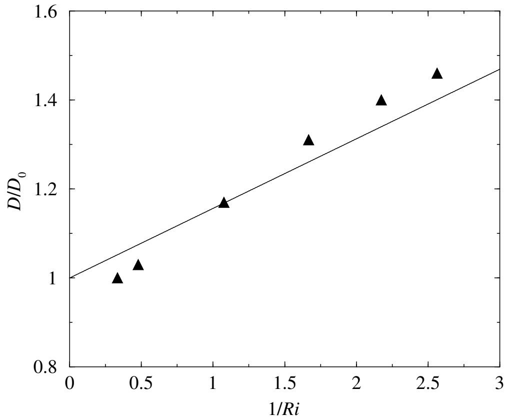
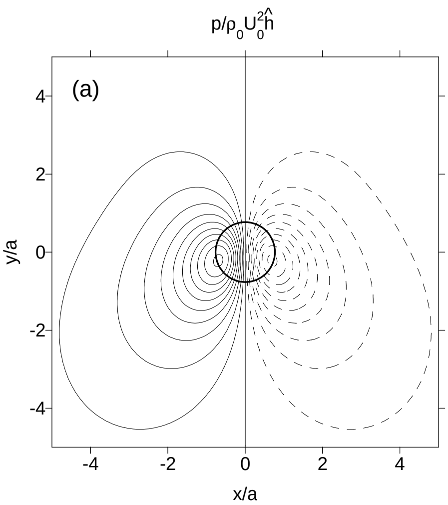


Figure 3. Normalized drag along *x* as a function of Ri^{-1} for the flow (13). Solid line: eq. (14), symbols: NH3D (from Valente 2000).

This surprising result is in agreement with the numerical simulations of Valente (2000) (fig. 3), where the NH3D model was used. The reason for this behavior is explained by fig. 4, which displays the pressure perturbation at the surface. This perturbation is stronger than in the constant wind case (Smith 1980), and the maxima and minima are displaced to the right of the surface wind, yielding no *y* drag component. This structure is in worse agreement with the numerical simulations than the previous case, which suggest that the *y* drag component may not be zero.



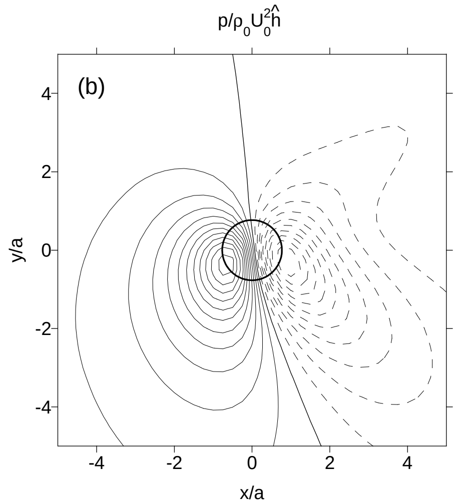


Figure 4. Normalized pressure perturbation at *Ri*=0.5 for the flow (13). Contours as in fig. 2.
(a) analytical model, (b) NH3D.

Nevertheless the main features of the flow are reproduced with some accuracy.

3.3. *Linearly decreasing wind – 2D ridge*

The previous results have been for an axisymmetric mountain. Now consider a linear wind profile over a 2D ridge:

$$U = U_0 - \alpha z. \tag{15}$$

The drag (9)-(10) reduces in that case to

$$D_x = D_{0x}\left(1 - \frac{1}{8Ri}\right), \tag{16}$$

where the definition of *Ri* is the same as in section 3.1. Equation (16) is asymptotically equal in the limit of large *Ri* to the corresponding result of Smith (1986) (his eq. (3.17)). The drag decreases as *Ri* decreases, as for an axisymmetric mountain, but the corresponding correction to D_{0x} is larger than that in (12) by a factor of 4/3, due to the absence of dispersion of the gravity waves.

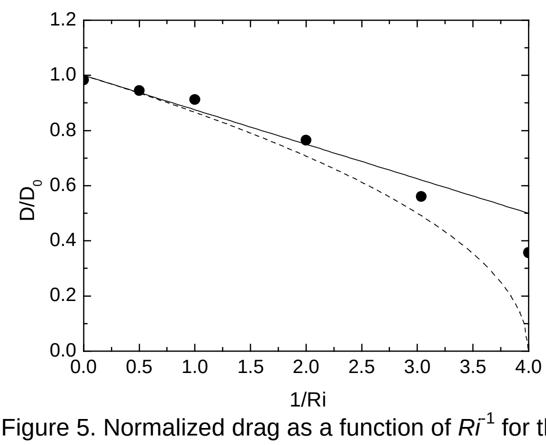


Figure 5. Normalized drag as a function of Ri^{-1} for the flow (15). Solid line: eq. (16), dashed line: Smith (1986), symbols: NH3D.

When the analytical prediction (16) is compared with results from the NH3D model, fairly good agreement is found (fig. 5).

The pressure perturbation at the surface, shown in fig. 6 for a bell-shaped ridge, also indicates that the flow structure is well reproduced by the analytical model, except for the lowest *Ri*, as would be expected.

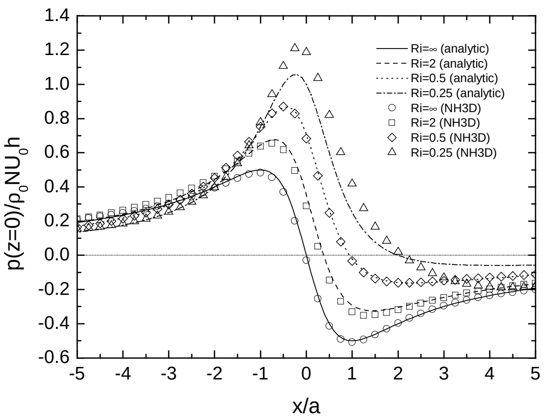


Figure 6. Normalized pressure perturbation at various values of *Ri* for the flow (15). Lines: analytical model, symbols: NH3D.

The pressure perturbation becomes progressively less anti-symmetric with respect to the ridge (whose peak is at *x*/*a*=0) as *Ri* decreases.

4. DISCUSSION

For the simple wind profiles considered in this study, it was found that the drag on an axisymmetric mountain or a 2D ridge varies proportionally to the inverse of the Richardson number of the flow. Whereas for a wind that varies linearly with height the drag decreases as *Ri* decreases, for a wind that turns with height maintaining its magnitude, the drag increases as *Ri* decreases. This explains previous numerical simulation results (Grubišić and Smolarkiewlcz 1997, Valente 2000). The present analytical model shows that the effect of a non-zero first derivative of the wind velocity at the surface is always to decrease the drag, while the effect of a negative second derivative (such as exists in a turning wind) is to increase the drag by a considerably larger amount. The model also predicts that the drag may not be aligned with the surface wind.

When results from the analytical model are compared with those obtained using a nonlinear, non-hydrostatic numerical model (albeit for approximately linear and hydrostatic conditions), good quantitative agreement is found, even for *Ri* of order one.

The analytical drag expressions derived for flow over a 2D ridge, although much shorter than those derived for an axisymmetric mountain, display qualitatively the same type of dependence on the first and second derivatives of the wind velocity and on *Ri*. However, the coefficients multiplying the corrections to the drag due to the wind variation are larger by a factor of 4/3, due to geometrical effects (the wind is forced to flow over a ridge, while it can go over or around a 3D mountain, causing wave dispersion).

In both cases, these corrections are independent of the exact shape of the orography, provided that this is axisymmetric or slab-symmetric. This feature adds much relevance to the present calculations.

The idea of this study was to isolate the effects of wind profile shear and curvature from nonlinear and non-hydrostatic effects. However, these latter effects certainly modify the results presented here in important ways, which deserve to be explored. Another outstanding problem is how to objectively define the height at which to take the wind velocity and its derivatives for use in the analytical formulas presented here from real atmospheric data, since the present model is inviscid, but in practice the wind tends to zero at the surface. These problems provide motivation for future investigations.

5. REFERENCES

Bacmeister, J. T. and Pierrehumbert, R. T., 1988: On the high-drag states of nonlinear stratified flow over an obstacle. *J. Atmos. Sci.*, **45**, 63-80.

Grisogono, B., 1994: Dissipation of wave drag in the atmospheric boundary layer. *J. Atmos. Sci.*, **54**, 1237-1243.

Grubišić, V. and Smolarkiewicz, P. K., 1997: The effect of critical levels on 3D orographic flows: linear regime. *J. Atmos. Sci.*, **54**, 1943-1960.

Miranda, P. M. A. and James, I. N. (1992) Non-linear three-dimensional effects on the wave drag: splitting flow and wave breaking. *Quart. J. Roy. Meteor. Soc.*, **118**, 1057-1081.

Shutts, G., 1995: Gravity wave drag parametrization over complex terrain: The effect of critical level absorption in directional wind shear. *Quart. J. Roy. Meteor. Soc.*, **121**, 1005-1021.

Smith, R. B., 1980: Linear theory of hydrostatic stratified flow past an isolated obstacle. *Tellus*, **32**, 348-364.

Smith, R. B., 1986: Further development of a theory of lee cyclogenesis. *J. Atmos. Sci.*, **43**, 1582-1602.

Teixeira, M. A. C., Miranda, P. M. A. and Valente, M. A. 2004: An analytical model of mountain wave drag for wind profiles with shear and curvature. *J. Atmos. Sci.*, **61**, 1040-1054.

Valente, M. A. 2000: Effects of directional wind shear on orographic gravity wave drag. PhD Thesis, University of Reading, 140pp.