6.19 A BAYESIAN ESTIMATE OF CT2 FROM SONIC ANEMOMETER DATA

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1. INTRODUCTION

We present a method of inferring the temperature structure parameter (CT2) from sonic anemometer data using a Bayesian analysis. Specifically, we first create a time series of temperature increments from the original sonic temperature time series and use the expected form of the temperature structure function to deduce the correlations between the increments in the series. By assuming that the increments are distributed according to a multivariate Gaussian distribution, the conditional probability for the increment time series for a given CT2 can be deduced. We then used Bayes' rule to find the conditional probability for CT2 given the observed increment time series and take the most probable value of CT2 as our measurement from the sonic anemometer data. Finally, we show a comparison of these CT2 values with those simultaneously obtained from a laser scintillometer.

2. THE METHOD

We wish to estimate the value of CT2 for a given sequence of *n* sonic temperature measurements $\vec{T} = [T_1, T_2, \ldots, T_n]$ equally spaced in time with intervals δ. We assume that the wind, *U*, is strong enough so that Taylor's hypothesis holds, and constant enough so that the measurements are equally spaced horizontally $d = U\delta$. We also assume that the temperature measurements have a negligible amount of noise. From this, we construct a time series of successive temperature increments $\vec{\Delta} = [T_{m+1} - T_1, T_{2m+1} - T_{m+1}, \ldots]$, where $m = \mathrm{ceil}(l_{\mathrm{m}} / d)$ is the minimum acceptable separation between measurements and $l_m = 30\ \mathrm{cm}$ is the approximate size of the sonic anemometer.

We also ignore the effect of the outer scale and assume that the structure function has the form,

$$\left\langle \left(T_{m+1} - T_1\right)^2 \right\rangle = C_T^2 d^{2/3} m^{2/3} \qquad (1)$$

This means that the variance of each temperature increment is

$$\left\langle \Delta_i^2 \right\rangle = C_T^2 d^{2/3} m^{2/3} = C_0 \qquad (2)$$

where we have assumed that the increments have zero mean. The structure function at a lag of *2dm* is simply the average of the square of the sum of two successive increments,

$$\left\langle \left(\Delta_i + \Delta_{i+1}\right)^2 \right\rangle = 2C_0 + 2C_1 = 2^{2/3} C_0 \qquad (3)$$

where $C_1 = \left\langle \Delta_i \Delta_{i+1} \right\rangle$ is the covariance of two successive increments, and is equal to

$$C_1 = \left(2^{-1/3} - 1\right) C_0 = -0.2063 C_0 . \qquad (4)$$

Knowing C_0 and C_1, and using the formula for the structure function at a lag of 3*dm*, we can deduce C_2, and then C_3 and so on until we have as many terms of the covariance function as for the temperature increment vector, $\vec{\Delta}$.

To find the conditional probability density, $\rho\left(\vec{\Delta} \mid C_T^2\right)$, it is convenient to use the increment correlations, $R_i = C_i / C_0$ and to define a correlation matrix.

$$\mathbf{R}_{ij} = R_{|i-j|} . \qquad (5)$$

From these we can obtain the multivariate Gaussian conditional probability,

$$\rho\left(\vec{\Delta} \mid C_T^2\right) = \frac{\exp\left[-B \middle/ \left(2 C_T^2 m^{2/3} d^{2/3}\right)\right]}{\left(2\pi C_T^2 m^{2/3} d^{2/3}\right)^{N/2} |\mathbf{R}|^{1/2}} \qquad (6)$$

where $|\mathbf{R}|$ is the determinant of the correlation matrix and *B* represents,

$$B = \vec{\Delta}' \mathbf{R}^{-1} \vec{\Delta} \qquad (7)$$

where $\mathbf{R}^{-1}$ is the matrix inverse of $\mathbf{R}$.

To determine $\rho\left(C_T^2 \mid \vec{\Delta}\right)$, we must choose an *a priori* probability distribution,

$$\rho\left(C_T^2\right) \propto \frac{1}{C_T^2} \qquad (8)$$

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It has the property of being flat in the logarithm of CT2, which is a desirable property given its dynamic range. Now, using Bayes' rule, we finally obtain

$$\rho\left(C_T^2 \mid \vec{\Delta}\right) = \frac{B^{N/2} \exp\left[-B\big/\left(2C_T^2 m^{2/3} d^{2/3}\right)\right]}{\Gamma\left(N/2\right)\left(2C_T^2 m^{2/3} d^{2/3}\right)^{N/2} C_T^2} \quad (9)$$

for which the most probable value of CT2 is

$$\tilde{C}_T^2 = \frac{B}{\left(N+2\right) m^{2/3} d^{2/3}}, \qquad (10)$$

and which we will take as our estimate.

3. MEASUREMENTS

Now we are ready to compare the CT2 values from our sonic anemometer data to those from a laser scintillometer. Our METEK sonic anemometer was placed over a flat test field 2.5 m Above Ground (AG) and took data for most of May and the beginning of June in 2003. A SINTEC SLS20 laser scintillometer accompanied the anemometer for most of that period. The SLS20 obtained estimates of the refractive index structure parameter (Cn2) at a wavelength of 670 nm over a path 185 m long at 1.6 m AG. However, due to several malfunctions of the SLS20 during the June period, only the May period is considered.

For a relatively dry atmosphere Cn2 is proportional to CT2, such that the SLS20 can serve as a norm for our CT2 estimates. The relationship is approximately,

$$C_T^2 \approx \frac{T^4}{A^2 P^2} C_n^2 \qquad (11)$$

where $A = 80$, *P* is the pressure in millibars, and *T* is the average air temperature in Kelvin. Furthermore, the temperature in Eq (11) is the actual temperature whereas the anemometer measures the sonic temperature, T_s, which depends on the specific humidity. We will assume that the atmosphere is dry enough so that these discrepancies are negligible. Finally, using an eddy-correlation method on the sonic anemometer wind and temperature data, we estimate the characteristic velocity scale, u_*, and temperature scale, T_*, of the surface layer. These scales allow us to perform a height adjustment of the SLS20 data using Monin-Obukhov similarity theory.

Time series of the CT2 estimates using the height-adjusted SLS20 Cn2 data from Eq (11), and the sonic anemometer data from the Bayesian method, are shown in Fig 1. It shows that the Bayesian estimates follow the scintillometer values reasonably well, except for a slight but persistent underestimation around midday and a persistent overestimation at night. The scatter plot in Fig 2 reveals this underestimation to be more or less constant along with some significant dispersion. The plot in Fig 2 also has a flat bottom. In other words, as the SLS20 estimates become smaller; less than 0.001, the Bayesian estimates reach a floor between 0.001 and 0.002. This may be due to the presence of noise in the sonic temperature, which dominates the temperature increments as CT2 becomes very small. Noise and other possible effects will be discussed in the next section.

4. DISCUSSION

As mentioned previously, noise in the sonic temperature measurements can affect the CT2 estimates. Fortunately, its effect can be taken into account if we know its variance, σ_T^2. Instead of using the previously derived increment covariance vector, $\vec{C}$, we use a modified vector that includes the noise contribution,

$$\vec{C}' = \vec{C} + \vec{\Gamma} \qquad (12)$$

where $\Gamma_0 = 2\sigma_T^2$, $\Gamma_1 = -\sigma_T^2$, and it is zero everywhere else, and use $\vec{C}'$ in the multivariate Gaussian distribution. Unfortunately, including noise makes a simple analytic solution impossible and requires some kind of numerical procedure.

Another factor that may affect the results is the presence of an outer scale. As mentioned previously, the noiseless increment covariance vector was deduced from a simple power-law structure function. However, if we have a good idea of the outer scale, L_o, (perhaps proportional to the height AG) and the functional form of the structure function that includes it, nothing prevents us from deducing a noiseless increment covariance vector that depends on it, $\vec{C}\left(L_o\right)$. If L_o and σ_T^2 are known, then we can evaluate $\rho\left(C_T^2 \mid \vec{\Delta}, \sigma_T^2, L_o\right)$. On the other hand, if we only know these additional factors approximately, through *a priori* distributions, $\rho\left(\sigma_T^2\right)$ and $\rho\left(L_o\right)$, then we can deduce $\rho\left(C_T^2, \sigma_T^2, L_o \mid \vec{\Delta}\right)$, requiring us to find the three most probable quantities.

Finally, there may be a problem with our assumption of a multivariate Gaussian distribution. Turbulent statistics are generally not Gaussian; however, the multivariate Gaussian distribution is a *maximum entropy* distribution, i.e. it is the flattest and broadest distribution possible that conforms to the first and second order statistics of the temperature increments. While there is no formal proof, experience has shown that Bayesian inference using such distributions tends to be robust with respect to such discrepancies (Box and Tiao, 1973).

5. CONCLUSIONS

To the best of the authors' knowledge, a Bayesian method for estimating CT2 from sonic anemometer data has been formulated for the first time in this work. It allows the possibility of using the power and flexibility of the Bayesian method to the problem of estimating CT2 in the surface layer. For instance, if there are other instruments present that give us some knowledge of what CT2 should be, then this knowledge may be incorporated into our method by using it as the *a priori* distribution $\rho\left(C_T^2\right)$ from which we derive the *a posteriori* distribution $\rho\left(C_T^2 \middle| \vec{\Delta}\right)$. In this way, we can obtain a CT2 estimate that combines all of our knowledge about it. Furthermore, the *a posteriori* distribution not only allows us to find a single representative value, such as the mode, mean or median, but also to find some kind of error estimate on that value, such as the standard deviation or a confidence interval.

Although the results from the Bayesian method follow the SLS20 estimates reasonably well, there is nevertheless a great deal of dispersion and a slight offset. More work remains to make this method fully mature by including such factors as the noise and the outer scale. This will undoubtedly make the procedure more numerical; however, it is the only way to obtain a comprehensive Bayesian method for data analysis in the atmospheric surface layer.

6. REFERENCES

Box, G. E. P., and G. C. Tiao, 1973: *Bayesian Inference in Statistical Analysis*. Addison-Wesley, 588 pp.

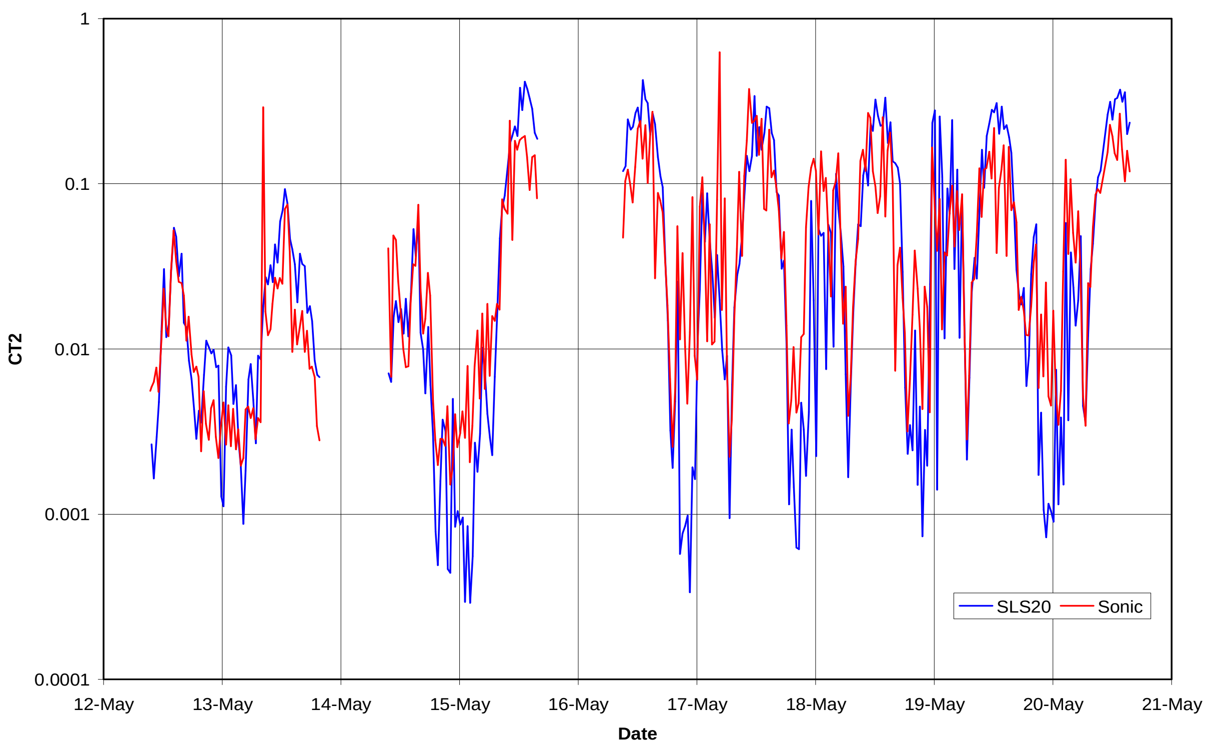

Figure 1. *Time series of the SLS20 (blue) and the sonic Bayesian (red) CT2 estimates for a period in May 2003.*

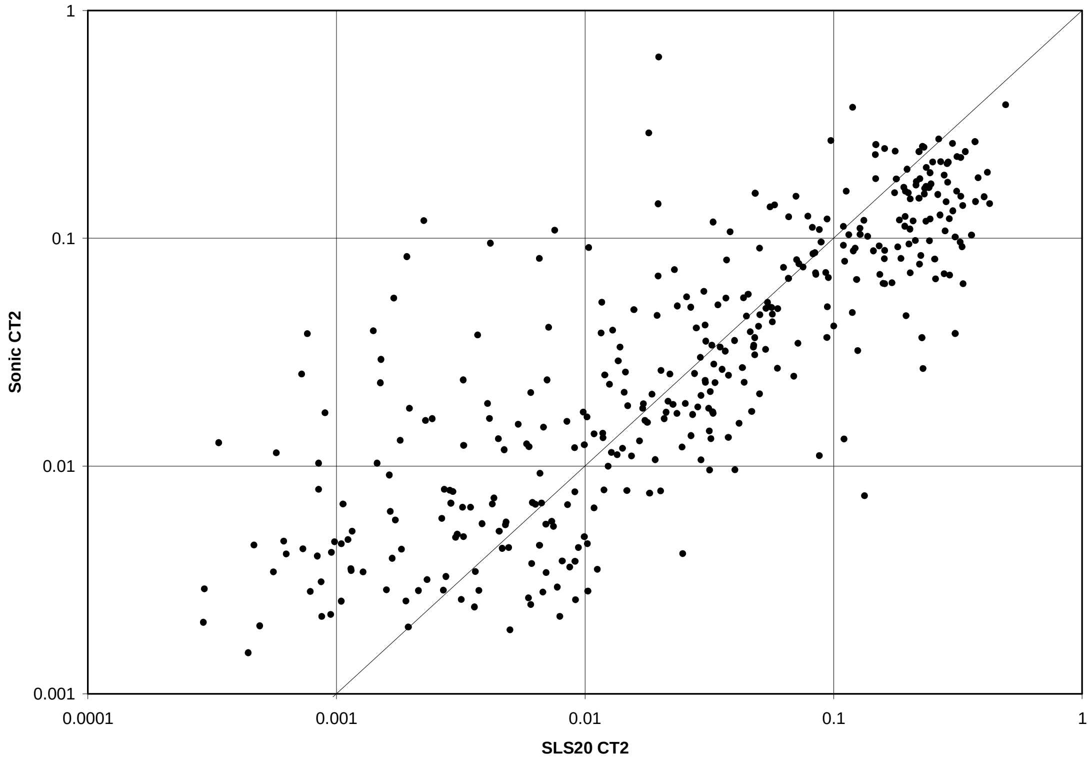

Figure 2. *A scatter plot of the sonic Bayesian versus the SLS20 scintillometer CT2 estimates.*