# THE INTRINSIC SENSITIVITY OF THE WAVE-MERGER PROBLEM¹ 

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## 1. INTRODUCTION

In meteorology, wave-merger (or trough merger) is defined as the amalgamation of two or more distinct $500-\mathrm{hPa}$ vorticity maxima. The most common form of wave-merger occurs when two separate vorticity centers interact with each other within a confluent background flow. Wave-merger cyclogenesis can bring intense storms to United States, but the occurrence of merger is very sensitive to initial conditions and background flow and is very difficult to predict.

In this paper, PV conservation is used to build the model. In the PV framework, the problem can be described as two PV anomalies moving in a confluent background flow. In order to simplify it, wave-merger is restricted to an inviscid, nondivergent barotropic atmosphere on a $f$ plane. Under these conditions the PV reduces to relative vorticity.

The two vortices are represented by rigid vortex patches. Assume each vortex localized is on patch which is a circle with radius $\eta$ of 100 km . Assume the vorticity of the northern center is a constant $\zeta_{10}$; the southern center is $\zeta_{20}$. Because we rearrange the vorticity in a smaller area (a circle with a radius $\eta$ ), then $\zeta_{10}$ and $\zeta_{20}$ are around $10^{-3} \mathrm{~s}^{-1}$, larger than that in the atmosphere .

The background flow is represented by a fixed hyperbolic deformation field. See Fig. 1.
The stream function is:

$$
\begin{equation*}
\psi_{b f}=-a x y \tag{1}
\end{equation*}
$$

$a$ is the strength of the confluent flow and it is positive and fixed at approximately $10^{-5} \mathrm{~s}^{-1}$.
Let $s=\frac{a}{\zeta_{10}+\zeta_{20}}$ be the ratio of the strength of the confluent flow over the sum of the strength of the two vortex centers.
The evolution of $\rho$ and $\theta$ (Fig. 1) in this problem can be described by the following nonlinear system:

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Fig.1. The background flow stream function , $\rho$ is the distance of the two vortex centers; $\theta$ is the azimuth measured in a clockwise manner between the line connecting the two centers and the positive direction of the confluent flow

$$
\left\{\begin{array}{l}
\frac{d \rho}{d t}=a \rho \cos (2 \theta)  \tag{2a}\\
\frac{d \theta}{d t}=-a \sin 2 \theta+\frac{\eta^{2}\left(\zeta_{10}+\zeta_{20}\right)}{2 \rho^{2}}
\end{array}\right.
$$

Let $\rho_{0}$ and $\theta_{0}$ be the initial values of $\rho$ and $\theta$. Then, $\rho_{0}$ and $\theta_{0}$ will determine the evolution of the system. An analytic method is used to find the relation between them, and the results from this relation are reported in the next section.

## 2. ANALYTIC RESULTS

1) In the nonlinear system, $\rho$ obtains its "local" minimum $\rho_{\min 45^{\circ}}$ at $\theta=45^{\circ}$ if the system turns clockwise and overpasses $45^{\circ}$.
2) $\rho$ has its "global" minimum $\rho_{\min 135^{\circ}}$ at $\theta=135^{\circ}$ if the system can turn counterclockwise and overpass $135^{\circ}$. The wave-merger events are more likely on $\theta=135^{\circ}$.
3) By deduction we can get an equation

$$
\begin{equation*}
1-x \sin 2 \theta_{0}=-\ln x \tag{3}
\end{equation*}
$$

and can show that when $\theta_{0} \in\left(0^{\circ}, 90^{\circ}\right)$, it has two solutions, the larger one $x_{2}\left(\theta_{0}\right)$ and smaller one $x_{1}\left(\theta_{0}\right)$. We draw $x_{2}\left(\theta_{0}\right)$ for $\left(45^{\circ}, 90^{\circ}\right)$; $x_{1}\left(\theta_{0}\right)$ for $\left(0^{\circ}, 45^{\circ}\right)$, see solid line in Fig.6.
4) When $\theta_{0} \in\left(45^{\circ}, 90^{\circ}\right)$,

If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}>x_{2}\left(\theta_{0}\right)$ at the initial time, then the system will turn clockwise and $\rho$ will obtain its "local" minimum $\rho_{\min 45^{\circ}}$ at $\theta=45^{\circ}$.
If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}<x_{2}\left(\theta_{0}\right)$ at the initial time, then $\rho$ will obtain its global minimum value $\rho_{\text {min135 }}$ at $\theta=135^{\circ}$.
If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}} \approx x_{2}\left(\theta_{0}\right)$ or $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}$ is near the solid line of Fig.6, the behavior of the system will be very sensitive, see Fig. 2.

If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}>x_{1}\left(\theta_{0}\right)$ at the initial time, then the
system can't turn counterclockwise to overpass $\theta=45^{\circ}$ and finally it will turn clockwise. $\rho$ will increase continuously, and there is no chance for merger. If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}<x_{1}\left(\theta_{0}\right)$, the system can turn counterclockwise to overpass $\theta=45^{\circ}$, and then it will turn counterclockwise to overpass $\theta=135^{\circ}$. $\rho$ will obtain its "global" minimum $\rho_{\min 135^{\circ}}$ at $\theta=135^{\circ}$ and merger could happen.
If $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}} \approx x_{1}\left(\theta_{0}\right)$ or $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}}$ is near the solid line of Fig.6, the behavior of the system will be very sensitive, see Fig. 3.
6) When $\theta_{0} \in\left(90^{\circ}, 135^{\circ}\right)$, the system is in the region with no sensitivity. It can only go counterclockwise and $\rho$ will obtain its "global" minimum $\rho_{\min 135^{\circ}}$ at $\theta=135^{\circ}$, no matter what $\rho_{0}$ is at the initial time.
5) When $\theta_{0} \in\left(0^{\circ}, 45^{\circ}\right)$,


Fig. 2. When $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}} \approx x_{2}\left(\theta_{0}\right)$, an observer on the northern vortex center will see the southern vortex turning clockwise and approaching to an unstable state point $\left(45^{\circ}, \sqrt{\frac{\eta^{2}}{2 s}}\right)$,the black dot. Then a very small difference in initial state $\left(\theta_{0}, \rho_{0}\right)$ can make the southern vortex go along totally different paths. The left path has lower chance for merger to occur, the right path has a higher chance.


Fig.3. If at the initial time the system has $\frac{2 s \rho_{0}{ }^{2}}{\eta^{2}} \approx x_{1}\left(\theta_{0}\right)$, then an observer staying on the northern vortex center will see the southern vortex is turning counterclockwise and approaching an unstable state point $\left(45^{\circ}, \sqrt{\frac{\eta^{2}}{2 s}}\right)$, the black dot. Then a very small difference in initial state $\left(\theta_{0}, \rho_{0}\right)$ can make the southern vortex go along different paths; the left path has no chance for merger to happen but the right path has chance for merger.

## 3. NUMERICAL SIMULATIONS

Set $a=2.5 \times 10^{-5} \mathrm{sec}^{-1}$,
$s=\frac{a}{\zeta_{10}+\zeta_{20}}=\frac{2.5 \times 10^{-5}}{10^{-2}}=\frac{1}{400}$ in these simulations.

For $\theta_{0} \in\left(45^{\circ}, 90^{\circ}\right)$, let $\theta_{0}=60^{\circ}$,
by solving equation(3), we can obtain $x_{2}(60)=1.8894$.
If $\rho_{0}=1.943 \times 10^{6} \mathrm{~m}$, then
$\frac{2 s \rho_{0}^{2}}{\eta^{2}}=1.8876<x_{2}\left(\theta_{0}\right)=x_{2}\left(60^{\circ}\right)=1.8894$,
$\rho$ will obtain its a minimum value $\rho_{\min 135^{\circ}}$ at $\theta=135^{\circ}$, the smallest distance is about 750 km see Fig. 4.
If $\rho_{0}=1.945 \times 10^{6} \mathrm{~m}$, then
$\frac{2 s \rho_{0}^{2}}{\eta^{2}}=1.8915>x_{2}\left(\theta_{0}\right)=x_{2}\left(60^{\circ}\right)=1.8894$,
$\rho$ will obtain its "local" minimum $\rho_{\min 45^{\circ}}$ at $\theta=45^{\circ}$, the smallest distance is about 1430 km , see Fig. 5 .

The only difference between Fig. 4 and Fig. 5 is 2 km error in $\rho_{0}$, the relative error is about $0.1 \%$. Because the "global" smallest distance is much smaller than the "local" one. The merger is more likely happen for case in which the system obtains its "global" smallest distance.

## 4. OBSERVATION STUDY

Now by using NCEP reanalysis data, the key values $\frac{2 s \rho_{0}^{2}}{\eta^{2}}$ of wave-merger cases are calculated. Gaza and Bosart (1990) studied 21 wave-merger cases over north America. We use them to make the calculation. In this work, the data 48,36 and 24 hours before the merger are used to calculate the key values on 48 hours before the merger; the data 36,24 and 12 hours before the merger are used to calculate the key value on 36 hours. Because of data availability, the key values on 48 and 36 hours before the merger can be obtained for 13 among the total 21 cases. The calculation result of the key value $\frac{2 s \rho_{0}^{2}}{\eta^{2}}$ are in Fig.6.


Fig.4. The solid line is for the time evolution of $\rho$; the dash line is for $\theta$ (radians). As $\theta_{0}=60^{\circ}$ and $\rho_{0}=1943 \mathrm{~km}$, then $\frac{2 s \rho_{0}^{2}}{\eta^{2}}=1.8876<x_{2}\left(60^{\circ}\right)=1.8894 \cdot \rho$ obtains its a minimum value $\rho_{\text {min135 }}$ at $\theta=135^{\circ}$, the smallest distance is about 750 km .


Fig.5. As $\theta_{0}=60^{\circ}$ and $\rho_{0}=1945 \mathrm{~km}$, then $\frac{2 s \rho_{0}^{2}}{\eta^{2}}=1.8915>x_{2}\left(60^{\circ}\right)=1.8894$.
$\rho$ obtains its a minimum value $\rho_{\min 45^{\circ}}$ at $\theta=45^{\circ}$, the smallest distance is about 1430 km .

For the 13 cases, on average, the initial states 48 hours before the merger are more sensitive than that 36 hours before the merger, leading to lower forecast skill.

This conclusion is consistent with the results from the operational LFM, which made two (seven) correct 48 (36) hour forecasts among 21 cases. For the samples of wrong 36 hour forecasts, the average $\left|\frac{2 s \rho_{0}^{2}}{\eta^{2}}-x\left(\theta_{0}\right)\right|$
is 2.45 , while for the correct 36 hours forecasts, it is 3.56. So, on average, the cases with wrong forecasts are more sensitive than that of the cases with correct forecasts. Certainly, there are many factors that can cause a poor model forecast. But the above results tell us that when the initial state is closer to the most sensitive region, the forecast is more likely to be wrong, as predicted by the theory.


Fig.6. The horizontal coordinate is for $\theta_{0}$; the vertical one is for the key value $\frac{2 s \rho_{0}^{2}}{\eta^{2}}$.
The solid line is the most sensitive region of the system, it is $x_{1}\left(\theta_{0}\right)$ when $\theta_{0} \in\left(0^{\circ}, 45^{\circ}\right)$; $x_{2}\left(\theta_{0}\right)$ when $\theta_{0} \in\left(45^{\circ}, 90^{\circ}\right)$. The black dots (red crosses) are the key values on 48 (36) hours before mergers. The crosses are farther from the most sensitive region (solid line) than the dots. This means that, on average, for these cases the initial states 48 hours before the mergers are more sensitive that the initial states 36 hours before the mergers.

## 5. DISCUSSIONS AND FUTURE WORK

By analytic method, the sensitivity mechanism and the most sensitive region of the wave-merger problem were found. When the initial state (initial conditions of the two vortices and the background flow) of the system is close to the most sensitive region, a very small difference in initial state can make the system evolve along completely different paths. Numerical simulations and observations support these results.

The wave-merger events can happen almost everywhere in the band of middle-latitude westerlies (Dean and Bosart, 1996).Therefore, the sensitivity in the wavemerger problem could cause trouble for any forecast,
especially for extended forecasts and climate forecasts. We can consider the wave-merger problem to be a sensitivity pattern (structure) in the atmosphere.

In order to use the methods reported here to improve the wave-merger forecast we need high spatial and temporal resolution data to determine the "accurate" positions of the two vortex centers. Conventional data can't provide enough information. Satellite data may provide advantages in this respect and should be investigated.

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