J6.8 INTERNAL WAVE TUNNELLING IN STRATIFIED SHEAR FLOWS

Geoffrey L. Brown*, J. Nault, and B. R. Sutherland
University of Alberta, Edmonton, Canada

1. INTRODUCTION

Internal gravity waves only propagate through fluids with density stratification. The maximum frequency of propagation is the buoyancy frequency, N, which under the Boussinesq approximation is defined as, $N^2 = -(g/\rho_0)d\bar{\rho}/dz$, where g is the gravitational acceleration and ρ_0 is a characteristic density of the fluid.

Theory predicts that an upward propagating internal wavepacket impinging on a layer of uniform density ($N = 0$) should reflect. A closer examination of this problem reveals that internal gravity waves become evanescent in unstratified regions so the amplitude decreases exponentially with height. Consider now a uniform-density layer of finite width, L, surrounded by stratified fluid. An incident wavepacket will partially reflect off the layer but will also partially transmit across the layer. This is the process of internal wave tunnelling.

Internal wave tunnelling between two ducts in the ocean has previously been described theoretically by Eckart (1961), who considered resonant energy transfer between different vertical modes of the main and seasonal thermocline in the ocean. Eckart's resonances in an atmospheric context were later described by Fritts & Yuan (1989) who also considered the effects of Doppler-shifting winds. Resonant theory is usefully applied to two-way energy transfer between ducts by low-order modes, however, it cannot describe one-way tunnelling of small-vertical scale internal waves from one duct to another.

Lindzen & Tung (1976) studied the reflection and over-reflection of internal gravity waves in the atmosphere but restricted themselves to mesoscale (hydrostatic) waves. Their model included a bottom boundary condition and a compressible, rather than Boussinesq, atmosphere. Primarily they analyzed low order modes.

An analytic theory for non-hydrostatic internal wave tunnelling through a weakly stratified fluid layer was derived by Sutherland & Yewchuk (2004). Here, that work is extended to include the effects of a shear layer coinciding with an unstratified region.

* *Corresponding author address*: Dept. Earth and Atmospheric Sciences, Univ. Alberta, Edmonton, AB, Canada T6G 2G1; e-mail: geoffrey.brown@ualberta.ca.

2. BACKGROUND THEORY

For simplicity, two-dimensional motions in a non rotating, inviscid, Boussinesq fluid are considered. Small perturbations about a steady state are assumed so that the governing equations may be linearized. Perturbations are assumed to vary as $\psi = \phi(z)\exp[i(kx - \omega t)]$, where k and ω are the horizontal wavenumber and frequency of the disturbance. The disturbances are known to satisfy the Taylor-Goldstein equation

$$\phi'' + k^2\left(\frac{N^2}{\Omega^2} + \frac{\overline{U}''}{\Omega} - 1\right)\phi = 0 \qquad (1)$$

where the Doppler-shifted frequency is defined as, $\Omega(z) \equiv \omega - k\overline{U}(z)$.

If the background velocity, $\overline{U}(z)$, and buoyancy frequency, $N^2(z)$, are smoothly varying functions, only in special cases can analytic solutions of (1) be found. In particular, if these profiles are piecewise-linear, such solutions are easily obtained.

$\overline{U}$ and N^2 are chosen supposing that the fluid is well-mixed in a shear layer of width L. Explicitly, we choose an unbounded shear layer

$$\overline{U}(z) = \begin{cases} U_0 & z > \frac{L}{2} \\ \frac{U_0}{L}(z + \frac{L}{2}) & |z| < \frac{L}{2} \\ 0 & z < -\frac{L}{2} \end{cases} \qquad (2)$$

and an "N^2-barrier" of the form

$$N^2(z) = \begin{cases} N_0^2 & |z| > \frac{L}{2} \\ 0 & |z| \le \frac{L}{2} \end{cases} . \qquad (3)$$

These are shown in Figure 1.

3. STABILITY SOLUTION

Before internal gravity wave tunnelling can be considered, the stability of the background flow itself must first be established.

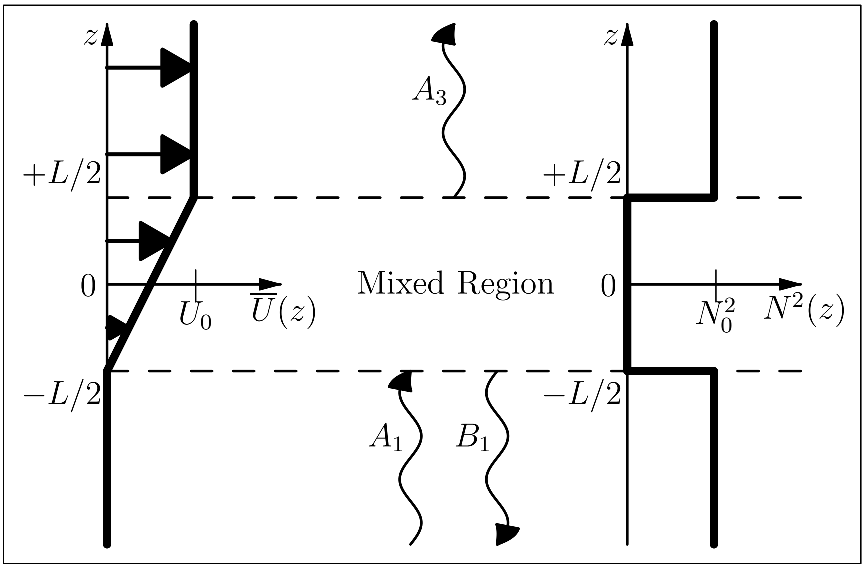

Figure 1: Background velocity and squared buoyancy frequency profiles defined for the tunnelling calculation.

For ease of analysis,we exploit symmetry by moving in a frame of reference at speed $U_0/2$ with respect to the wind below $z = -L/2$. The solutions to (1) take the form

$$\phi(z) = \begin{cases} A_3 e^{-i\gamma_3 z} & z > \frac{L}{2} \\ A_2 e^{-kz} + B_2 e^{kz} & |z| < \frac{L}{2} \\ B_1 e^{i\gamma_1 z} & z < -\frac{L}{2} \end{cases} \tag{4}$$

with

$$\gamma_1 = k\left[\frac{N_0^2}{\Omega_1^2} - 1\right]^{1/2} \tag{5}$$

and

$$\gamma_3 = k\left[\frac{N_0^2}{\Omega_3^2} - 1\right]^{1/2}. \tag{6}$$

Here, $\Omega_1 = \omega + kU_0/2$ and $\Omega_3 = \omega - kU_0/2$ are Doppler-shifted frequencies. Because γ_1 and γ_3 represent vertical wavenumbers for $z > |L/2|$, they are defined so that disturbances flow outward from the shear layer. When γ_1 and γ_3 are complex, branch cuts are taken so that disturbances decay exponentially away from the shear layer.

Matching conditions at $z = \pm L/2$ are found by requiring that the vertical velocity and pressure is continuous (Drazin and Reid 1981). For a fluid with continuous background density, $\bar{\rho}(z)$, and continuous $\overline{U}(z)$ this amounts to requiring that

$$\Delta[\phi] = 0 \text{ and } \Delta[(\omega/k - \overline{U})\phi' + \overline{U}'\phi] = 0. \tag{7}$$

On applying the matching conditions (7) to the solution for ϕ in (4) the eigenvalue relation

$$\tilde{\omega}^6 + C_4\tilde{\omega}^4 + C_2\tilde{\omega}^2 + C_0 = 0 \tag{8}$$

with $C_i(\tilde{k}; \mathsf{Ri})$ is obtained. Here, $\tilde{\omega} \equiv \omega(L/U_0)$ and $\tilde{k} \equiv kL$ are dimensionless and, $\mathsf{Ri} \equiv N_0^2/(U_0/L)^2$ is the bulk Richardson number. There are three roots for $\tilde{\omega}^2$ in this equation but only one is physically possible, the other two are spurious. The appropriate root is determined by taking limit as $N_0 \to 0$ and selecting that which corresponds with the classic result for unstratified shear flow (Drazin and Reid 1981):

$$\tilde{\omega}^2 = [(\tilde{k} - 1)^2 - e^{-2\tilde{k}}]/4. \tag{9}$$

The physical root is decomposed into its real and imaginary parts, $\tilde{\omega} \equiv \tilde{\omega}_r + i\tilde{\omega}_i$, and is plotted in Figure 2. The top plot shows that the system is unstable for every value of Ri. However, the instability only occurs for a small range of kL for any Ri and the growth rate of the instability, $\tilde{\omega}_i$, decreases as Ri increases.

Therefore, for large enough Ri, the instability growth rate will be smaller than the time for wave propagation across the mixed region. So it is not unreasonable to consider internal wave tunnelling in these circumstances.

4. TUNNELLING SOLUTION

Now consider a wave of amplitude A_1, incident from below on a shear layer. For $\overline{U}$ and N^2 profiles

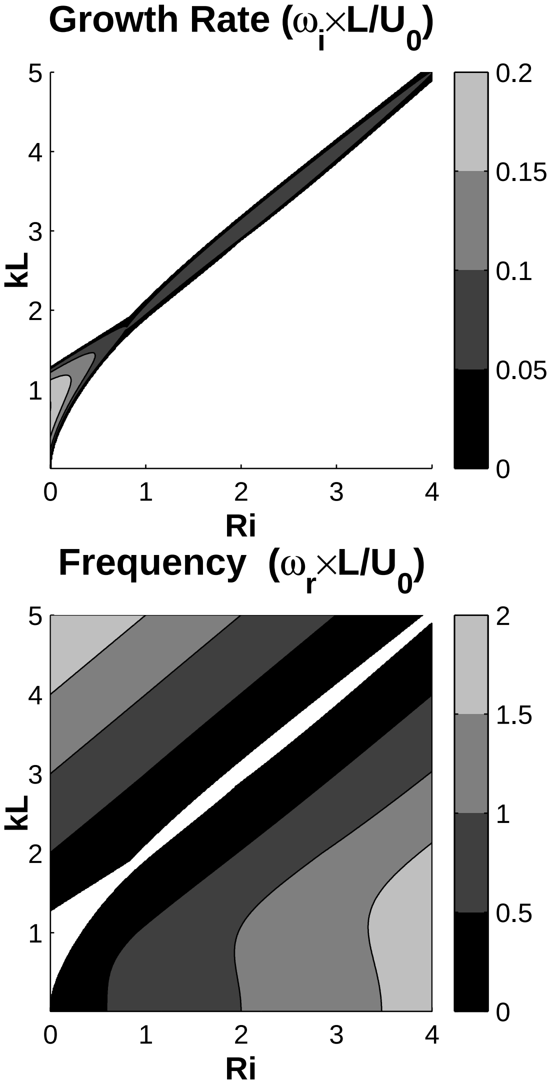

Figure 2: The contour plots show the non-dimensional growth rate and frequency defined by the physical root of (8).

as in (2) and (3), the solution to (1) becomes

$$\phi(z) = \begin{cases} A_3 e^{-i\Gamma_3 z} & z > \frac{L}{2} \\ A_2 e^{-kz} + B_2 e^{kz} & |z| < \frac{L}{2} \\ A_1 e^{-i\Gamma_1 z} + B_1 e^{i\Gamma_1 z} & z < -\frac{L}{2} \end{cases} \tag{10}$$

where Γ_1 and Γ_3 are the same as γ_1 and γ_3 in (5) and (6) except that Ω_1 is replaced by ω and Ω_3 is replaced by $\omega - kU_0$ since we are now back in the frame of reference moving with the flow below $z = -L/2$. Applying the matching conditions (7) to the above solution gives a system of four equations and five unknowns. Solving for the transmitted amplitude, A_3, in terms of the incident amplitude, A_1, gives a transmission coefficient, $T = |A_3/A_1|^2$, which represents the fraction of energy transported. T is a function of $\tilde{k}$, Θ and, Ri, where $\Theta \equiv \cos^{-1}(\omega/N_0)$, which represents the angle at which lines of constant phase are oriented from the vertical. In the limit of infinitesimally small shear ($U_0 \to 0$, Ri $\to \infty$), Sutherland and Yewchuk (2004) showed that the transmission coefficient is

$$T = \left[1 + \left(\frac{\sinh(\tilde{k})}{\sin 2\Theta}\right)^2\right]^{-1}. \tag{11}$$

For finite Ri, Figure 3 plots values of T as a func-

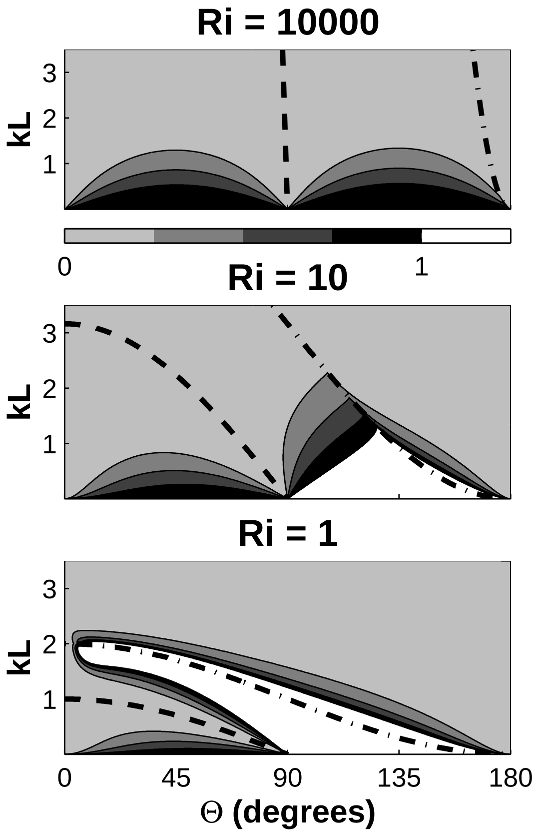

Figure 3: The contour plots show the values of the transmission coefficient, T, for internal gravity waves traversing an N^2 barrier for several values of Ri. Θ is the angle of propagation of the incident waves with respect to the vertical. At the dashed line $c = \omega/k = U_0$ and at the dashed-dot line $|\omega - kU_0| = N_0$.

tion of kL and Θ in three cases, each with different values of Ri.

In the first plot $\mathsf{Ri} = 10000$, which approximates the case where $U_0 = 0$, T approaches the result (11).

For smaller Ri, a new over-transmission ($T > 1$) regime appears. Along the dashed line the phase speed of incident waves is exactly U_0 and above this line (larger kL) internal waves encounter a critical level. Along the dashed-dot line the frequency of incident waves, when Doppler-shifted by the background flow, equals N_0 ($|\omega - kU_0| = N_0$).Waves with still larger kL are evanescent and transmission drops to zero.

5. CONCLUSIONS

We derived an analytic prediction for internal wave tunnelling through a shear layer surrounded by stratified fluid. For weak shear, maximum transmission occurs for waves with $\Theta = 45°$ (equal horizontal and vertical wavenumbers). For stronger shear, the waves over-transmit (meaning the incident waves extract energy from the shear and transmit at a larger amplitude). Peak over-transmission occurs when $kL = (1+\omega/N_0)\sqrt{\mathsf{Ri}}$ and no transmission occurs when $kL = (\omega/N_0)\sqrt{\mathsf{Ri}}$.

Future work will examine the transmission and reflection of internal gravity waves for more complex and more realistic background profiles of N^2 and $\overline{U}$, and include anelastic and finite-amplitude effects.

Acknowledgements

This research was supported by the Canadian Foundation for Climate and Atmospheric Science (CFCAS).

References

Drazin, P. G. and Reid, W. H. 1981. *Hydrodynamic Stability*. Cambridge University Press, Cambridge, England.

Eckart, C. 1961. Internal waves in the ocean. *Phys. Fluids*, 4:791–799.

Fritts, D. C. and Yuan, L. 1989. An analysis of gravity wave ducting in the atmosphere: Eckart's resonances in thermal and Doppler ducts. *J. Geophys. Res.*, 94(D15):18455–18466.

Lindzen, R. S. and Tung, K. K. 1976. Banded convective activity and ducted gravity waves. *Monthly Weather Review*, 104(12):1602–1617.

Sutherland, B. R. and Yewchuk, K. 2004. Internal wave tunnelling. *J. Fluid Mech.*, 511:125–134.