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Solving for pressure : acceleration and accuracy

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1 Introduction

Meso and large scale models tend nowadays to be non-hydrostatic. Time step restrictions due to fast waves are alleviated by considering semi-implicit schemes (Thomas et al, 2003) or by use of the anelastic approximation (Meso-NH model, Lafore et al, 1998, or Clark, 1977 among others); in both cases an elliptic equation has to be solved. This equation is typically ill conditioned, due to the range of scales considered and the ratio of horizontal to vertical scales; it is also non-separable, as the domain is not parallepipedic, due to the presence of orography; in most discretizations it is also found non-symmetric. Thomas et al (2003) make a survey of the different methods to solve this equation; they adopt the preconditioning proposed in Bernardet (1995), hereafter B95 and rely on generalizations of the conjugate gradient method to non-symmetric problems such as GCR(k).

In B95 it was noticed that the divergence in the continuity equation and the gradient of pressure are naturally adjoint operators, but, if the divergence is naturally discretized in flux form, two discretizations of the pressure gradient come naturally, and moreover the elliptic equation might be discretized directly without reference to the gradient or divergence operators, with a minimum use of averaging for example. With a certain placement of the quantities defining the metrics of the computational grid, it was found that the discretizations of the pressure terms are equivalent (outside boundaries), but that disposition was not used in the Clark (1977) model nor retained in the Meso-NH model.

We therefore seek in this paper to convince the reader that a symmetric elliptic equation can be obtained by an adequate design of the extrapolations at the boundary for the discretized gradient and divergence operators; the Helmholtz equation will then be solved at a minimal cost; for example, the orthomin solver is adequate for non-symmetric problems but necessitates two applications of the preconditioner per iteration, instead of one for the standard conjugate gradient.

Clark (2003) suggested to examine the truncation error for the pressure term as a test for consistency of the model. We will show that, with our discretization, this error has a simple analytic expression and we will compare it to the result with other formulations. As Clark (2003) has used a very smooth orography, we will design a more stringent test for the comparison. One should note that these truncation errors may give a yardstick for convergence of pressure.

Many authors (Sundqvist,1976, Lin, 1997, Janjic, 1977, Mahrer, 1984, and in oceanography Haney, 1991, Mellor et al, 1994) have discussed pressure gradient errors linked to steep bottom slope in terrain following coordinates ; D. Dempsey (1998) give a account of their amplitude and effect after dissipation of acoustic waves in the context of a non-hydrostatic compressible model; here, with an anelastic model, pressure is not the primary variable; it is its gradient which gives the reaction force necessary to enforce the anelastic constraint, so pressure force error can be seen more directly. Pressure force at the meso-scale is determined as the solution of a global problem, so local considerations of pressure gradient errors are inappropriate. So we will use the same test cases as Dempsey (1998) to investigate error sources for the pressure term.

2 Model description

The analytic equations of the Lipps-Hemler (1982) system are standard and we follow the conventions of the Meso-NH model (Lafore et al, 1998):

$$
\begin{aligned}
\frac{d\mathbf{u}}{dt} &= -\rho\nabla p + \mathbf{g}\frac{\theta'}{\overline{\theta}} && (1)\\
\frac{d\theta}{dt} &= 0 \\
\nabla.\overline{\rho}\mathbf{u} &= 0
\end{aligned}
$$

Placement of variables for an Arakawa C-grid is as shown:

	w	
u, U, x, d_{xx}	$p, \overline{\rho}, J$	u
ζ, d_{xz}	w, W, z, θ, d_{zz}	

with the horizontal (vertical) velocity u (w), the contravariant velocities U (W), the potential temperature θ and its deviation θ' from the reference $\overline{\theta}(z)$, the reference density $\overline{\rho}(z)$ Here the geometry of the grid is defined by the value of x (z) defined at the u (w) points with[1] $d_{xx} = \overline{\delta_x x}^x$, $\left(d_{zz} = \overline{\delta_z z}^z\right)$, $d_{xz} = \delta_x z$, $J = \delta_x x \delta_z z$, in contrast with B95 where x, z were defined at the ζ point. $\overline{\rho}$ $\left(\overline{\theta}\right)$ is the density (potential temperature) of the reference state, p is related to perturbation pressure and $\widetilde{\rho} = J\overline{\rho}$ is the Jacobian weighted density. At the bottom (lateral) boundaries the normal velocity is defined as the outermost velocity point; we need also extra pressure points at the boundary. With these definitions, the pressure gradient in non-conservative form as advocated in B95 and as they appear in the Meso-NH model write :

$$
\begin{aligned}
p_{fx} &= -\frac{\overline{\widetilde{\rho}}^x}{d_{xx}}\delta_x p + \frac{\overline{\widetilde{\rho}}^x}{d_{xx}} d_{zx}\overline{\overline{\left(\frac{\delta_z p}{d_{zz}}\right)}^x}^z && (2)\\
p_{fz} &= -\frac{\overline{\widetilde{\rho}}^z}{d_{zz}}\delta_z p && (3)
\end{aligned}
$$

where $\widetilde{\rho} = \overline{\rho}J$, p_{fx}, p_{fz} represent the pressure forces and *buoy* the buoyancy forces. The form of the gradient is devised so that it is in adjoint relation with the divergence operator of the continuity equation: this equation and the relation between contravariant velocities and Cartesian velocities are :

$$
\begin{aligned}
0 &= J\begin{cases} -\left.\overline{\widetilde{\rho}}^z W\right|_{z=h} \\ \delta_x\left(\overline{\widetilde{\rho}}^x U\right) + \delta_z\left(\overline{\widetilde{\rho}}^z W\right) \\ \left.\overline{\widetilde{\rho}}^z W\right|_{z=h} \end{cases}\\
\overline{\widetilde{\rho}}^x U &= \overline{\widetilde{\rho}}^x u/d_{xx} \\
\overline{\widetilde{\rho}}^z W &= \left(\overline{\widetilde{\rho}}^z w - \overline{\overline{\left(\overline{\widetilde{\rho}}^x \hat{u}/d_{xx}\right)}^z d_{zx}}^x\right)/d_{zz}
\end{aligned}
$$

Here the divergence is augmented by the boundary conditions to match the number of pressure points; we shall see their role in making it an adjoint of the non-conservative gradient operator for some scalar products.

We notice that some of the calculus we have done above necessitates some extrapolation of the inner u velocities; they are marked with a hat.

The so-called conservative form of the gradient effectively preserves momentum and is deduced from the divergence formula when we remark that the horizontal unit vector $\mathbf{i}$ has a null divergence, so that $\mathbf{i}.\nabla p = \nabla.\mathbf{i}p$; discretized, it gives the following momentum equations, where p has a slightly different meaning :

$$
\begin{aligned}
p_{fx} &= -\frac{\delta_x}{d_{xx}}p - \frac{\delta_z}{\overline{d_{zz}}^z}\left(\frac{d_{zx}}{d_{xx}}\overline{p}^{xz}\right) && (4)\\
p_{fz} &= -\frac{1}{d_{zz}}\delta_z p && (5)
\end{aligned}
$$

3 A symmetric pressure equation :

A symmetric pressure problem can be solved iteratively by the celebrated congugate gradient method (CG); the CG method can be slightly generalized by the use a non-trivial scalar product, widening the class of problems where it can be employed. Let A^t denote the transpose of a matrix A and M be the diagonal matrix representing the scalar product. A^* is the adjoint of A iff for any p, q

$$
\begin{aligned}
\langle Ap; q\rangle &= \langle p; A^*q\rangle \\
p^t A^t M q &= p^t M A^* q \\
A^* &= M^{-1}A^t M
\end{aligned}
$$

[1] $\delta_x\alpha = \alpha(x + \Delta x/2) - \alpha(x - \Delta x/2)$ and $\overline{\alpha}^x = 1/2\left[\alpha(x + \Delta x/2) + \alpha(x - \Delta x/2)\right]$ are the Schuman (1962) operators

and similarly composition of operators gives $(AB)^* = B^*A^*$. Here, the operator to consider is the divergence

$$\nabla . \left(\begin{array}{c} \rho u \\ \rho w \end{array} \right) = JDC \left(\begin{array}{c} \overline{\overline{\rho}}^x u \\ \overline{\overline{\rho}}^z w \end{array} \right)$$

D is the "flat" divergence, C is the transformation to contravariant velocities; C itself comprises averaging and extrapolation operators; scalar products are employed for pressure and u, w components of velocity to find the adequate adjoint. Usually, if A^t is a consistent discretization of an operator, it will not be so of A^* when we change the scalar product. We adjust the scalar products so that each of the operators composing the divergence has an adjoint with a physical meaning. As shown in appendix, the difficulty arises from the averaging-extrapolation operators and is solved by adjusting the scalar product on w, by using a simple extrapolation of u by copy and by considering that the boundary pressure point is collocated with the boundary w point. The outcome is a different pressure at the boundary than from the standard discretization, but it is easy to verify that the pressure gradient is unchanged.

4 Truncation errors:

The inner product of (1) and $\overline{\rho}\mathbf{u}$ result in the kinetic energy equation

$$\frac{\partial \overline{\rho} \mathbf{u}.\mathbf{u}}{2\partial t} + \nabla . \left[\overline{\rho} \mathbf{u} \left(\frac{\mathbf{u}.\mathbf{u}}{2} + p \right) \right] = \frac{g}{\overline{\theta}} w \theta'$$

With $q = \frac{1}{2} \left(\overline{\overline{\overline{\rho}}^x u^2}^x + \overline{\overline{\overline{\rho}}^z w^2}^z \right)$ it can be discretized as

$$\frac{\partial (q)}{\partial t} + \Gamma_q = -\delta_x \left(U \overline{p}^x \right) - \delta_z \left(W \overline{p}^z \right) + buoy \quad (6)$$

$$\Gamma_q = \delta_x \left(U \overline{q}^x / 2 \right) + \delta_z \left(W \overline{q}^z / 2 \right)$$

is estimated from eqn.2,**??** multiplied by u, w and averaged

$$\frac{\partial q}{\partial t} + 2\Gamma_a = \overline{u p_{fx}}^x + \overline{w p_{fz}}^z \quad (7)$$

It can be shown that many cancellations occur in the truncation error for advection $\Lambda_a = \Gamma_a - \Gamma_q$, resulting in a third order quantity, and that the truncation error for the pressure force

$$\Lambda_p = \overline{u p_{fx}}^x + \overline{w p_{fz}}^z - \delta_x \left(U \overline{p}^x \right) - \delta_z \left(W \overline{p}^z \right)$$

is third order as well; when the "conservative" pressure force is used (eqn.4,5), no such cancellations occur.

Such a strong dependence upon wave-length of the truncation error has led us to make a test with a high wave-number orography. So to compare the amplitude of the pressure truncation error with the two formulations (fig.1 and fig.2), we have used a periodic channel with sinusoidal wave-number 1 orography with 7 points in the horizontal and the vertical with $\Delta x = \Delta z = 2km$; the pressure term is the one which added to a uniform velocity $u = 1, w = 0$ makes the velocity to conform with the anelastic constraint and boundary conditions; results are expressed as the ratio of the error to the maximum of pressure work, and are of the order of 3%. Errors decrease rapidly when the orography is better resolved, from 30% for 5 points per wave-length to 0.6% for 9 points.

The numerical calculations distinguish the two schemes,but confirm the result of Clark (2003) that, although the B95 scheme is more accurate, no great difference results; the truncation error for the D98 pressure gradient introduced in the next section has a lower but comparable accuracy, as shown in Fig.3.

5 Accuracy of the pressure gradient

As mentioned in the introduction, pressure gradient error in a stratified atmosphere has been a major concern for a long time; Dempsey (1998), hereafter D98 reexamines the problem in a mesoscale atmospheric setting; we reproduce his test. A witch of Agnesi profile mountain is considered with a half width $a = 2\Delta x = 4km$ and height $h_0 = 2km$, thus making a slope 1/2 at mid-height; an atmosphere at rest with constant stratification $N = 0.01s^{-1}$ above $z = 1km$ and $N = 0.02s^{-1}$ under is used, thus the dependence of the pressure perturbation under $z = 1km$ is quadratic; vertical mesh size is $\Delta z = 100m$ far from the mountain.

"Standard" horizontal pressure gradient used by D98 is determined as follows: on each side of a

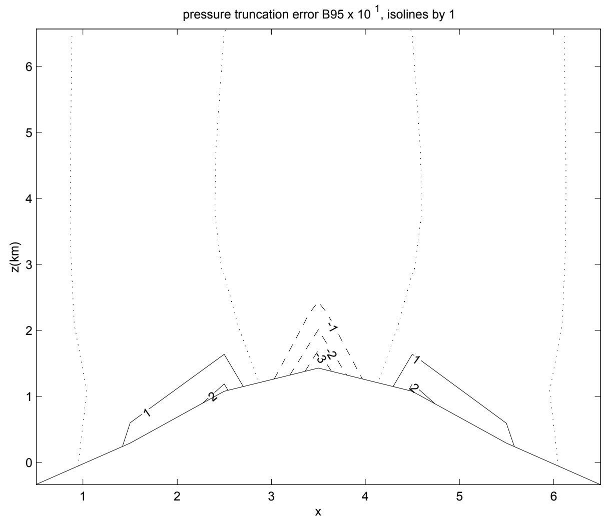

Figure 1: Relative truncation error for pressure according to the B95 discretization; values $\times 10$; 2.7%quadratic mean error

u velocity point where horizontal gradient is to be determined, pressure is first calculated by linear interpolation at the altitude of the u point; then the pressure gradient is determined from the right and left interpolated pressures; when the u-point is under the sloping orography, linear extrapolation is used instead; thus the scheme is rather well suited to the description of a quadratic departure from the reference pressure. Note that the "standard" scheme we have examined involves linear interpolations, in contrast to the "Mahrer scheme" which uses quadratic interpolations (Mahrer, 1984).

Results show that the B95 (fig.4) and C87 schemes have comparable accuracy, and the D98 scheme has half that error; projection of the pressure gradient to match the boundary conditions lower the error; for example the B95 error is cut by a factor of 3 (fig.5), but the projected D98 error is null, thus the behavior of an anelastic model and a compressible one is quite different.

6 Summary and conclusion

Three schemes for the calculation of the pressure gradient were examined: the formulation of Clark (1977,2003), the proposition of Dempsey (1998)

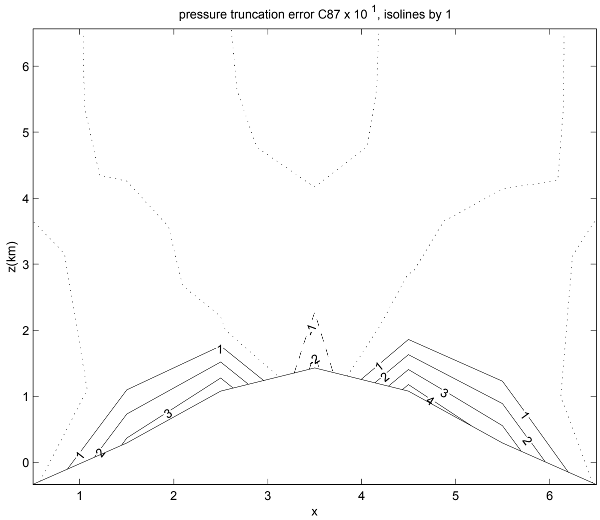

Figure 2: same as Fig.1 for the C87 discretization; 2.9% quadratic mean error

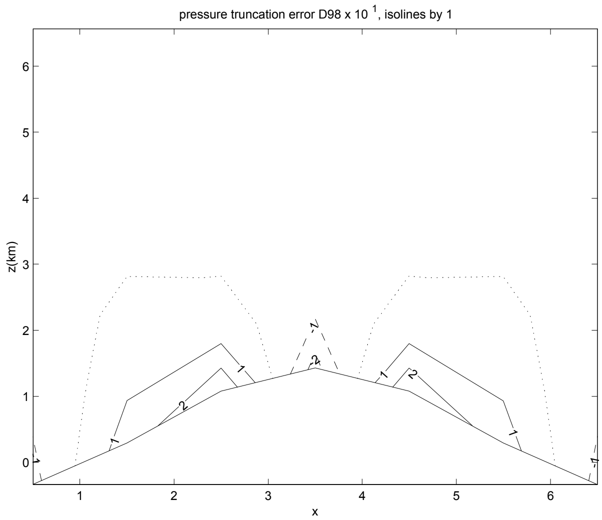

Figure 3: Same as Fig.1 for D98 discretization; 3.4% quadratic mean error

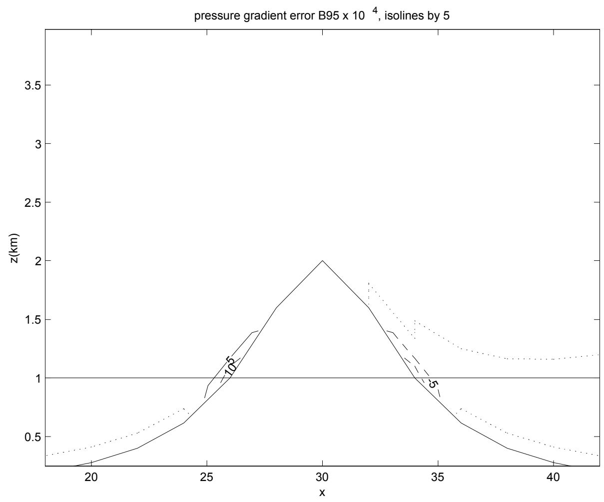

Figure 4: Horizontal pressure gradient error according to B95

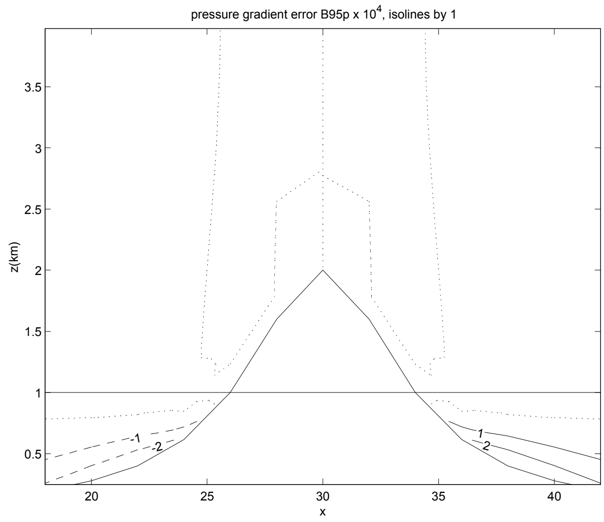

Figure 5: Horizontal pressure gradient error according to B95 after projection of gradient to match the anelastic and boundary conditions

and the one from Bernardet (1995).

To get a symmetric problem for pressure, we had to take for the gradient operator the adjoint of the divergence; the gradient has to be consistent, so we had to adjust the scalar product defining the adjoint and the extrapolation present in the divergence when forming the contravariant velocities. It happens that the adequate extrapolation is a mere copy to the outside velocity point; it seems a general fact that the extrapolations needed to get a symmetric problem are less accurate than the inner scheme.

Accuracy of the schemes can be seen through the calculation of the truncation error, which is the departure from the discrete energy conservation. Although we can formally determine the truncation error for the B95 scheme (Appendix B), numerical tests show that the three schemes have a comparable accuracy: truncation error is of the order of 3% for a seven points per wave-length orography.

On the test-problem set up by Dempsey, the D98 scheme has half the error of the other schemes; surprisingly, after enforcing on the gradient the anelastic and boundary conditions, the D98 scheme has a null error, in contrast with its behavior in an elastic model. We should remark that standard tests of the accuracy of a model are made with an isothermal atmosphere flowing over the mountain, thus the vertical stratification is constant, at the difference of the test of Dempsey.

We should mention the clever two-time level technique of Clark (2003) as a means to avoid iterative methods; it appears to be a very economic solution for leap-frog time-stepping, and has to be tested in semi-implicit models with long time steps.

The paper was presented in a 2-D periodic channel; however the soundness of the approach has been tested in a three dimensional box without further complications;extension to elastic models is straightforward.

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