P4R.2 CORRECTIONS TO AND CONSIDERATIONS OF THE SPECTRUM WIDTH EQUATION

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1. INTRODUCTION

The application of the spectrum width, i.e. $\sqrt{\overline{\sigma_v^2(\vec{r})}}$, has been focused on isolating the contribution due to turbulence from other mechanisms. Most successful applications of $\sqrt{\overline{\sigma_v^2(\vec{r})}}$ are obtained for vertically, or nearly vertically, pointed radar (e.g. Brewster and Zrnic 1986, Hitschfeld and Dennis 1956, Fukao etc. 1994, Kollias and Albrecht 1999, Rogers and Tripp, 1964); for horizontally scanning radars like the WSR-88D, only a few cases are reported (Istok and Doviak 1986, Fang, Doviak and Melnikov 2003, Meischner etc. 2001). Both the weather signal and turbulent velocity are random variables. In order to correctly interpret and relate them to each other, one has to rely on the accuracy of the estimated spectrum widths due to the various spectral broadening mechanisms. The relation (Doviak and Zrnic 1993, Eq. 5.67)

$$\sigma_v^2(\vec{r}_0) = \sigma_0^2 + \sigma_\alpha^2 + \sigma_t^2 + \sigma_s^2 + \sigma_d^2$$

is often used equation in radar meteorology. Here, σ_0, σ_α, σ_t, σ_s and σ_d represents the expected spectrum width due to hydrometer's oscillation and/or wobbling, antenna rotation, turbulence, shear and diversity of the fall speeds of hydrometers. However, according to the author's knowledge, there has not been detailed derivation for this relation. It is not clear under what condition(s) this relation is valid. Furthermore, this relation states that the square of the expected total spectrum width equals to the sum of the square of the expected widths associated with each mechanism. There are no cross terms. Existing theories don't show cross terms, or the conditions under which cross terms can be ignored. One of purposes of this study is to rigorously derive relations for the first and second moment of the expected Doppler spectrum, and make clear under what condition(s) the above equation is valid.

Each mechanism contributing to the spectrum width broadening has been studied and the associated theories were developed based upon different conceptual models, such as that for radar antenna rotation (Doviak and Zrnic 1993), shear (Sloss and Atlas 1968, Atlas, Srivastava and Sloss 1969, Doviak and Zrinc 1993) and beam width (Atlas 1964, Nastron 1997, Chu 2002). By decomposing the measured radial velocity into steady, turbulent and terminal velocities, we attempt to put all contributors in one single unified analytical theoretical frame. This is another motivation of this study.

2. Autocorrelation Function and Spectrum Width

For the sake of developing some basic formulas for application to other sections, we begin our analysis with the weather signal's autocorrelation function, $R(mT_s, \vec{r}_0)$ assuming antenna rotation. The starting expression can be written as (i.e. Eq. (5.59a) of Doviak and Zrnic 1993)

$$R(mT_s, \vec{r}_0) = \sum_k E\left[A_k^*(0) A_k(mT_s) F_k^*(0,\vec{r}_0) F_k(mT_s,\vec{r}_0) e^{-j4\pi v_k mT_s/\lambda}\right] \quad (1),$$

where A_k is the complex amplitude of voltage echoed from the k^{th} particle, F_k the weight imposed on k^{th} particle due to the radar beam pattern. " * " in (1) denotes the complex conjugate, and E denotes the expectation operation. $\vec{r}_0$, locates the center of V_6, and is a function of time if antenna is rotating. Consider an elemental volume dV in which there are Δk hydrometers. The weather signals from dV have the correlation function $dR(mT_s)$

$$dR(mT_s, \vec{r}_0, \Delta k) = \sum_k^{k+\Delta k} \left|F_k(0,\vec{r}_0)\right|^2 \rho_{kF}(mT_s) R_k(0) \rho_{ko}(mT_s) E\left(e^{-j4\pi v_k mT_s/\lambda}\right) \quad (2)$$

where

$$\rho_{kF}(mT_s, \vec{r}_0) \equiv \frac{F_k^*(0,\vec{r}_0) F_k(mT_s,\vec{r}_0)}{\left|F(0,\vec{r}_0)\right|^2} \quad (3)$$

$$\rho_{ko}(mT_s) \equiv \frac{E\left[A_k^*(0) A_k(mT_s)\right]}{E\left[\left|A_k(0)\right|^2\right]} \quad (4)$$

are spatially dependent correlation coefficients associated with beam displacement due to antenna rotation.

3. Spectrum Width with Stationary Antenna

For a stationary antenna radar, $\rho_{kF}(mT_s, \vec{r}_0) \equiv 1$ if the scatterers' displacements in mT_s are small compared to the beamwidth as it typically is for meteorological scatterers. The subscript k locates the elemental volume dV where Δk hydrometers are located. In general $\rho_{ko}(mT_s)$ is a function of the location of Δk,

$$\rho_o(mT_s, \vec{r}) \equiv \frac{E\left[A^*(0,\vec{r}) A(mT_s,\vec{r})\right]}{E\left[\left|A(0,\vec{r})\right|^2\right]} \quad (5)$$

where $\vec{r}$ replaces the subscript k that locates Δk and dV. Thus Eq. (2) can be expressed as

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$$
\begin{aligned}
&dR(mT_s,\vec{r}_0,\vec{r}) \\
&= dR(0,\vec{r}_0,\vec{r})\rho_o(mT_s,\vec{r})E\left[e^{-j4\pi v(\vec{r})mT_s/\lambda}\right]
\end{aligned}
\tag{6}
$$

where $\rho_o(mT_s,\vec{r})$ is the correlation coefficient associated with hydrometers' oscillation and/or wobbling at point $\vec{r}$ and integrating in space then one has

$$
\begin{aligned}
&R(mT_s,\vec{r}_0) \\
&= \int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})\rho_o(mT_s,\vec{r})E\left(e^{-j4\pi v(\vec{r})mT_s/\lambda}\right)dV
\end{aligned}
\tag{7}
$$

According to Reynolds decomposition (Pope 2003), the velocity field $v(\vec{r},t)$ can be decomposed into time averaged mean (i.e. steady) $v_s(\vec{r})$ and turbulent $v_t(\vec{r},t)$ components. If the radar beam axis is elevated above the horizon, $v(\vec{r},t)$ should also contain a component $v_{dp}(\vec{r})$ due to the terminal velocity of the hydrometer at $\vec{r}$. If turbulence is approximately statistically stationary at least during the period of radar sampling, Eq. (7) then can be rewritten as

$$
\begin{aligned}
&R(mT_s,\vec{r}_0) \\
&= \int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})\rho_o(mT_s,\vec{r})\rho_d(mT_s,\vec{r})\rho_t(mT_s,\vec{r})\rho_s(mT_s,\vec{r})dV
\end{aligned}
\tag{8}
$$

where

$$\rho_d(mT_s,\vec{r}) = E\left[e^{-j4\pi v_{dp}(\vec{r})mT_s/\lambda}\right] \tag{9}$$

$$\rho_t(mT_s,\vec{r}) = E\left[e^{-j4\pi v_t(\vec{r},t)mT_s/\lambda}\right] \tag{10}$$

$$\rho_s(mT_s,\vec{r}) = e^{-j4\pi v_s(\vec{r})mT_s/\lambda} \tag{11}$$

are the point correlation coefficients associated with spread of terminal velocity of hydrometers, turbulence, and steady flow at $\vec{r}$. The weather signal's correlation coefficient $\rho(mT_s,\vec{r}_0)$ defined as

$$\rho(mT_s,\vec{r}_0) \equiv \frac{R(mT_s,\vec{r}_0)}{R(0,\vec{r}_0)} \tag{12}$$

is then

$$
\begin{aligned}
&\rho(mT_s,\vec{r}_0) \\
&= \frac{\int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})\rho_d(mT_s,\vec{r})\rho_o(mT_s,\vec{r})\rho_t(mT_s,\vec{r})\rho_s(mT_s,\vec{r})dV}{\int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})dV}, \\
&= \int_V H_n(\vec{r}_0,\vec{r})\rho_o(mT_s,\vec{r})\rho_d(mT_s,\vec{r})\rho_t(mT_s,\vec{r})\rho_s(mT_s,\vec{r})dV,
\end{aligned}
\tag{13}
$$

where

$$H_n(\vec{r}_0,\vec{r}) \equiv \frac{I(\vec{r}_0,\vec{r})\eta(\vec{r})}{\iiint I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (\mathrm{m}^{-3}) \tag{14}$$

The Doppler spectrum in the frequency domain is defined as the Fourier transform of autocorrelation function of weather signal. Using the property of Fourier transform of $\tilde{F}[g_1g_2g_3g_4] = \tilde{F}[g_1]*\tilde{F}[g_2]*\tilde{F}[g_3]*\tilde{F}[g_4]$ and noting that the volumetric integration and Fourier transform are commutative, one obtains the weighted normalized expected Doppler spectrum of weather signals in the velocity domain

$$
\begin{aligned}
&\bar{S}_n(v,\vec{r}_0) = \tilde{F}[\rho(\tau,\vec{r}_0)], \\
&= \int_V H_n(\vec{r}_0,\vec{r})\tilde{F}[\rho_o(mT_s,\vec{r})]*\tilde{F}[\rho_d(mT_{s0},\vec{r})]*\tilde{F}[\rho_t(mT_s,\vec{r})]*\tilde{F}[\rho_s(mT_s,\vec{r})]dV, \\
&= \int_V H_n(\vec{r}_0,\vec{r})S_{no}(v,\vec{r})*S_{nd}(v,\vec{r})*S_{nt}(v,\vec{r})*S_{ns}(v,\vec{r})dV.
\end{aligned}
\tag{15}
$$

The first moment $v_m(\vec{r}_0)$ of the weighted normalized Doppler spectrum is then

$$
\begin{aligned}
&v_m(\vec{r}_0) = \int_{-\infty}^{\infty} v\bar{S}_n(v,\vec{r}_0)dv \\
&= \int_{-\infty}^{\infty} v\int_V H_n(\vec{r}_0,\vec{r})S_{no}(v,\vec{r})*S_{nd}(v,\vec{r})*S_{nt}(v,\vec{r})*S_{ns}(v,\vec{r})dVdv \\
&= \overline{v_o(\vec{r})} + \overline{v_d(\vec{r})} + \overline{v_t(\vec{r})} + \overline{v_s(\vec{r})}, \\
&= \overline{v_d(\vec{r})} + \overline{v_s(\vec{r})},
\end{aligned}
\tag{16}
$$

where $v_i(\vec{r}) = \int_{-\infty}^{\infty} vS_{ni}(v,\vec{r})dv$, *i* = *o*, *d*, *t* or *s*, is the first moment of $S_i(v,\vec{r})$. In this study, we assume $v_o(\vec{r}) = 0$. Dovaik and Zrnic (1993) shows that $S_{nt}(v,\vec{r})$ is equal to the probability density function of turbulent flow and its first moment, i.e. $v_t(\vec{r})$, is equal to zero. It can be shown that $S_{ns}(v,\vec{r})$ is a delta function and its first moment is just the velocity of steady flow $v_s(\vec{r})$. These results have been used in Eq. (16). The above equation shows that the first moment of the weighted normalized expected Doppler spectrum is the sum of the first moments of each individual normalized spectrum weighted by radar beam pattern, range weighting and reflectivity.

The second central moment of the weighted normalized expected Doppler spectrum, can be written as

$$\overline{\sigma_v^2(\vec{r})} = \int_{-\infty}^{\infty}[v - v_m(\vec{r}_0)]^2\int_V H_n(\vec{r}_0,\vec{r})S_{no}(v,\vec{r})*S_{nd}(v,\vec{r})*S_{nt}(v,\vec{r})*S_{ns}(v,\vec{r})dVdv \tag{17}$$

It can be shown

$$
\begin{aligned}
&\overline{\sigma_v^2(\vec{r})} = \int_V H_n(\vec{r}_0,\vec{r})\left[\sigma_o^2(\vec{r}) + \sigma_d^2(\vec{r}) + \sigma_s^2(\vec{r}) + \sigma_t^2(\vec{r})\right]dV \\
&= \overline{\sigma_o^2(\vec{r})} + \overline{\sigma_d^2(\vec{r})} + \overline{\sigma_{\tilde{d}}^2(\vec{r})} + \overline{\sigma_s^2(\vec{r})} + \overline{\sigma_t^2(\vec{r})} + 2\overline{\left\{v_d(\vec{r}) - \overline{v_d(\vec{r})}\right\}\left\{v_s(\vec{r}) - \overline{v_s(\vec{r})}\right\}}
\end{aligned}
\tag{18}
$$

It is noteworthy that this relation differs from Eq. 5.67 given by Doviak and Zrnic (1993) under the condition of a stationary radar antenna in three aspects. Firstly, all terms in Eq. (41) are the volumetric mean weighted by $H_n(\vec{r}_0,\vec{r})$; secondly, there is a new term associated with the gradient of the mean terminal velocity across the radar beam; finally, there is another new term depending on the cross product of the gradient of the mean terminal velocity and wind. Both new terms are have heretofore been neglected in the literature. If the spectrum of the terminal velocity or reflectivity is uniform across the radar beam, then both new terms will disappear in Eq.(18). Furthermore, if hydrometer's oscillation/wobbling and turbulence are also locally homogeneous (i.e., over V_6), the velocity variances associated with these mechanisms can be removed from the volumetric integration and the corresponding over bars can be removed. Thus Eq. (18) reduces to

$$\sigma_v^2(\vec{r}_0) = \sigma_o^2(\vec{r}_0) + \sigma_t^2(\vec{r}_0) + \overline{\sigma_s^2(\vec{r})} + \sigma_d^2(\vec{r}_0) \quad (19).$$

Although there is no over bar (i.e. weighted volumetric mean) over any term in the Eq. 5.67 given by Doviak and Zrnic, considering their Eq. 5.51, their Eq. 5.67 should be identical to the Eq. (19) given here. From the derivation given above, it is clear that expression (19) is valid under the conditions: 1. the backscatter sections of hydrometers are independent of turbulence; 2. the influence of both steady and turbulent flows on the weighting function $F_k(mT_s, \vec{r}_0)$ is negligible; 3. oscillation/wobbling of hydrometer is locally homogeneous; 4. turbulence is locally homogeneous; 5. the spectrum of the terminal velocity of hydrometers or the reflectivity is locally homogeneous. It should be pointed out that when we derive the expressions of first and second central moments of the weighted normalized expected Doppler spectrum we assume the velocity field is stationary and therefore $v_s(\vec{r})$ is steady. However, this assumption is not a necessary condition for our derivation. If $v_s(\vec{r})$ is not only a function of space but also a function of time, Eqs. (16) and (18) are still available.

4. The Effect of Radar Antenna Rotation on Spectrum Width

If radar antenna is stationary, section 3 shows that the square of the total spectrum width equals to the weighted sum of the square of spectrum width associated with each contributor. This section extends the discussions in section 3 to scanning radars. For a scanning radar Eq. (1) can be rewritten as

$$R(mT_s, \vec{r}_0) = \sum_k E\left[A_k^*(0) A_k(mT_s) F_k^*(\vec{r}_{00}) F_k(\vec{r}_0) e^{-j4\pi v_k mT_s/\lambda}\right] \quad (20)$$

where $\vec{r}_{00} = (r_{00}, \theta_{00}, \phi_{00})$ locates the center of the resolution volume V_6 at zero lag, and, because V_6 can be displaced $d\vec{r}_0$ in mT_s, $\vec{r}_0 = \vec{r}_{00} + d\vec{r}_0$. Under the condition that the antenna pattern is product separable in the θ, φ directions, from Eq. (5.40) in Doviak and Zrnic (1993)

$$F_k(0, \vec{r}_{00}) = \frac{\sqrt{C} f^2(\theta_k - \theta_{00}) f^2(\phi_k - \phi_{00}) W_s(r_k, r_{00})}{l(\vec{r}_k) r_k^2} \quad (21)$$

$$F_k(mT_s, \vec{r}_0) = \frac{\sqrt{C} f^2(\theta_k - \theta_{00}) f^2(\phi_k + \Delta\phi - \phi_{00}) W_s(r_k, r_{00})}{l(\vec{r}_k) r_k^2} \quad (22)$$

$$C = \frac{P_t g^2 \lambda^2}{(4\pi)^3} \quad (23)$$

where $\Delta\phi = \alpha m T_s$, the one-way transmission loss, P_t the power transmitted by radar, g the gain of antenna, λ the length of electromagnetic waves transmitted by radar. Assume the $A_k(mT_s)$ does not depend on v_k, and consider an elemental volume $dV = r^2 \sin\theta dr d\theta d\phi$, where θ is the zenith angle. The auto covariance for signals backscattered from Δk hydrometers in dV can be written as

$$dR(mT_s, \vec{r}_0, \vec{r}) = \rho_F(\vec{r}, \vec{r}_0, d\vec{r}_0) \rho_o(mT_s, \vec{r}) E\left(e^{-j4\pi v(\vec{r}) mT_s/\lambda}\right) I(\vec{r}_0, \vec{r}) \eta(\vec{r}) dV \quad (24).$$

It follows that $R(mT_s, \vec{r}_0)$ is the integral of Eq. (25). That is

$$R(mT_s, \vec{r}_0) = \int_V \rho_F(\vec{r}, \vec{r}_0, d\vec{r}_0) \rho_o(mT_s, \vec{r}) \rho_d(mT_s, \vec{r}) \rho_t(mT_s, \vec{r}) \rho_s(mT_s, \vec{r}) I(\vec{r}_0, \vec{r}) \eta(\vec{r}) dV \quad (26).$$

In general the antenna radiation pattern $f^2(\theta) f^2(\phi)$ is a complex function, but typically where $f^2(\theta) f^2(\phi)$ has significant value the pattern function can be well represented by a real Gaussian function. Thus noting the subscript k can be replaced by $\vec{r}$ in the arguments of the variables, one obtains

$$\rho_F(\Delta\phi, \vec{r}_0, \vec{r}) = e^{-\frac{2(\phi - \phi_0)\Delta\phi + \Delta\phi^2}{4\sigma_\theta^2} \cos^2\theta_e} \quad (27)$$

where $\Delta\phi = \alpha m T_s = m\delta\phi_0$ is the azimuth displacement of radar beam in mT_s. $\rho_F(\Delta\phi, \vec{r}_0, \vec{r})$ is only a function of the spatial coordinates φ, φ_0, $\theta_e = \pi/2 - \theta_0$ where θ_0 is the zenith angle and σ_θ^2 is the second central moment of the two-way power pattern for a circularly symmetrical beam.

Weather radar process a number M of received signal samples to reduce the uncertainty in the estimates of spectral moments. Antenna motion (usually azimuth rotation), combined with processing M weather signal samples, produce an effective beam broadened in azimuth. To estimate the first and second moments using PPP, the correlation function $R^{(i)}(T_s, \vec{r}_{0m})$ (i.e., the products of $V^*(m)V(m+1)$) of continuous weather signal samples are summed, and the moments are calculated from the summed correlation function at the output of an integrator $\vec{r}_{0m}$ is the location of the resolution volume V_6 during the reception of the 2nd weather signal sample of the mth contiguous pair.

The correlation function $R^{(i)}(T_s, \vec{r}_{0m})$, for the mth sample pair at the input to the integrator,

$$R^{(i)}(T_s,\vec{r}_{0m})=\int_V \rho_F(\vec{r},\vec{r}_{0m},d\vec{r}_0)\rho_o(T_s,\vec{r})\rho_d(T_s,\vec{r})\rho_t(T_s,\vec{r})\rho_s(T_s,\vec{r})I(\vec{r}_{0m},\vec{r})\eta(\vec{r})dV \tag{28}$$
$$=\int_V \rho_F(\vec{r},\vec{r}_{0m},d\vec{r}_0)\rho_{odst}(T_s,\vec{r})I(\vec{r}_{0m},\vec{r})\eta(\vec{r})dV$$
$$=\int_0^{2\pi} f^4(\phi-\phi_{0m})e^{-\beta(\phi-\phi_{0m})}\iint \rho(\delta\phi_0)\rho_{odst}(T_s,\vec{r})\frac{C|W_s(r,r_{0m})|^2}{l^2(\vec{r})r^4} f^4(\theta-\theta_0)r^2\cos\theta_e\eta(\vec{r})drd\theta d\phi$$

is the expected correlation function of a pair weather signals echoed by hydrometeors having a cross section weighted by an antenna pattern that has shifted $\delta\phi_0=\alpha T_s$ to φ_{0m} in T_s, and $\beta=\frac{\delta\phi_0\cos^2\theta_e}{2\sigma_\theta^2}$, $\rho(\delta\phi_0)=e^{-\frac{\delta\phi_0^2\cos^2\theta_e}{4\sigma_\theta^2}}=e^{-\frac{\delta\phi_0^2}{4\sigma_\phi^2}}$, $\rho_{odst}(T_s,\vec{r})=\rho_o(T_s,\vec{r})\rho_d(T_s,\vec{r})\rho_t(T_s,\vec{r})\rho_s(T_s,\vec{r})$ and $\sigma_\phi=\frac{\sigma_\theta}{\cos\theta_e}$. The integrator has an impulse response $h(t)$, and noting that integration and expectation are commutative, its expected output is obtained by convolving $R^{(i)}(t)$ with $h(t)$; that is,

$$R^{(o)}(T_s,\vec{r}_{0m})=h(t)*R^{(i)}(T_s,\vec{r}_{0m}) \tag{29}$$
$$=\int_0^{2\pi} I_{w1}(\phi)\iint \rho(\delta\phi_0)\rho_{odst}(T_s,\vec{r})\frac{C|W_s(r,r_{0m})|^2}{l^2(\vec{r})r^4} f^4(\theta-\theta_0)r^2\cos\theta_e\eta(\vec{r})drd\theta d\phi$$

where

$$I_{w1}(\phi) \equiv \int_{-\infty}^{t=\phi_{0/\alpha}} h(t-\tau)*\left[f^4(\phi-\alpha\tau)e^{-\beta(\phi-\alpha\tau)}\right]d\tau \tag{30}$$

The correlation coefficient at the receiver output is defined as

$$\rho^{(o)}(T_s,\vec{r}_0)=\frac{R^{(o)}(T_s,\vec{r}_0)}{R^{(o)}(0,\vec{r}_0)} \tag{31}$$

where $R^{(o)}(T_s,\vec{r}_0)$ is the sum of M + 1 expected power samples. Substitution of Eq. (77), with T_s = 0, for the denominator of Eq. (78), produces

$$\rho^{(o)}(T_s,\vec{r}_0)\equiv\rho^{(o)}(\phi_{00})=\rho(\delta\phi_0) \tag{32}$$
$$=\frac{\int_0^{2\pi} I_{w1}(\phi)\iint \rho(\delta\phi_0)\rho_{odst}(T_s,\vec{r})\frac{|W_s(r,r_{0m})|^2}{l^2(\vec{r})r^4} f^4(\theta-\theta_0)r^2\cos\theta_e\eta(\vec{r})drd\theta d\phi}{\int_0^{2\pi} I_{w2}(\phi)\iint \frac{|W_s(r,r_{0m})|^2}{l^2(\vec{r})r^4} f^4(\theta-\theta_0)r^2\cos\theta_e\eta(\vec{r})drd\theta d\phi}$$

where

$$I_{w2}(\phi)\equiv\int_{-\infty}^{t=\phi_{0/\alpha}} h(t-\tau)*\left[f^4(\phi-\alpha\tau)\right]d\tau \tag{33}$$

and

$$f^4(\phi-\alpha\tau)=e^{-\frac{(\phi-\alpha\tau)^2}{2\sigma_\phi^2}} \tag{34}$$

is a azimuthal weighting function and, for circularly symmetric radiation patterns, $\sigma_\theta=\sigma_\phi\sin\theta_0$ (Doviak and Zrnic, 1993, Serction 5.3 errata) has been substituted for σ_θ.

Consider a finite-time, MT_s, un-weighted block integration because both spectral and PPP algorithms in weather radars typically employ it. The impulse response is given by,

$$h(t)=\begin{cases}1/MT_s & if \quad 0\le t\le MT_s \\ 0 & otherwise\end{cases} \tag{35}$$

It can be shown that

$$\rho^{(o)}(T_s,\vec{r}_0)=e^{-\frac{1}{8}\left(\frac{\delta\phi_0}{\sigma_\phi}\right)^2}\int_V H_{en}(\vec{r}_0',\vec{r}_0,\vec{r})\rho_{odst}(T_s,\vec{r})dV \tag{36}$$

where

$$H_{en}(\vec{r}_0',\vec{r}_0,\vec{r})=\frac{f_e^4(\phi-\phi_0')\frac{|W_s(r,r_0)|^2}{l^2(\vec{r})r^4}f^4(\theta-\theta_0)\eta(\vec{r})}{\int_V f_e^4(\phi-\phi_0)\frac{|W_s(r,r_0)|^2}{l^2(\vec{r})r^4}f^4(\theta-\theta_0)\eta(\vec{r})dV} \tag{37}$$

It is noteworthy that, for a horizontally scanning radar, $H_{en}(\vec{r}_0',\vec{r}_0,\vec{r})=H_{en}(\vec{r}_0,\vec{r})$ because $\phi_0'\equiv\phi_0-\frac{\delta\phi_0}{2}$.

If we have multiple complex multipliers and integrators, at each output of them we obtain a correlation coefficient with the different lag time. For lag time mT_s, by changing T_s to MT_s in Eq. (36), one has

$$\rho^{(o)}(mT_s,\vec{r}_0)=e^{-\frac{1}{8}\left(\frac{m\delta\phi_0}{\sigma_\phi}\right)^2}\int_V H_{en}(\vec{r}_0,\vec{r})\rho_{odst}(mT_s,\vec{r})dV \tag{38}$$
$$=\rho_\alpha(mT_s)\int_V H_{en}(\vec{r}_0,\vec{r})\rho_{odst}(mT_s,\vec{r})dV$$
$$=\rho_\alpha(mT_s)\int_V H_{en}(\vec{r}_0,\vec{r})\rho_o(mT_s,\vec{r})\rho_d(mT_s,\vec{r})\rho_s(mT_s,\vec{r})\rho_d(mT_s,\vec{r})dV$$

and

$$\rho_\alpha(mT_s)=e^{-\frac{1}{8}\left(\frac{\alpha mT_s}{\sigma_\phi}\right)^2} \tag{39}$$

under the condition that $m < M$. The normalized Doppler spectrum associated with radar rotation is

$$S_{n\alpha}=\tilde{F}[\rho_\alpha(mT_s)]=\tilde{F}\left[e^{-\frac{1}{8}\left(\frac{\alpha t}{\sigma_\phi}\right)^2}\right] \tag{40}$$
$$=\sqrt{\frac{2\pi\theta_1^2}{\alpha^2\lambda^2\cos^2\theta_e\ln 2}}e^{-\frac{2\pi^2\theta_1^2v^2}{\alpha^2\lambda^2\cos^2\theta_e\ln 2}}$$

and the analytical expressions of the normalized Doppler spectrum for other mechanisms are just same as that given in section 3.

Fourier transforming Eq. (38) and following the procedures used in section 3, we can show that the mean velocity and the spectrum width obtained from a weighted normalized Doppler spectrum for a scanning radar are

$$v_m(\vec{r}_0)=\overline{v_\alpha(\vec{r})}+\overline{v_o(\vec{r})}+\overline{v_d(\vec{r})}+\overline{v_t(\vec{r})}+\overline{v_s(\vec{r})} \\ =\overline{v_d(\vec{r})}+\overline{v_s(\vec{r})} \qquad (41)$$

and

$$\overline{\sigma_v^2(\vec{r})}=\int_V H_{en}(\vec{r}_0,\vec{r})\left[\sigma_o^2(\vec{r})+\sigma_o^2(\vec{r})+\sigma_d^2(\vec{r})+\sigma_s^2(\vec{r})+\sigma_t^2(\vec{r})\right]dV \qquad (42)$$
$$=\sigma_\alpha^2+\overline{\sigma_o^2(\vec{r})}+\overline{\sigma_d^2(\vec{r})}+\overline{\sigma_{\tilde{d}}^2(\vec{r})}+\overline{\sigma_s^2(\vec{r})}+\overline{\sigma_t^2(\vec{r})}+2\overline{\{v_d(\vec{r})-\bar{v}_d(\vec{r}_0)\}\{v_s(\vec{r})-\bar{v}_s(\vec{r}_0)\}}$$

It is easy to show that the first and second central moments associated with radar rotation are

$$\overline{v_\alpha(\vec{r})}=0 \qquad (43)$$

and

$$\sigma_\alpha^2=\left(\frac{\alpha\lambda\cos\theta_e\sqrt{\ln 2}}{2\pi\theta_1}\right)^2 \qquad (44)$$

which is not the function of space.and identical to the square root of Eq. (C.23) given by Doviak and Zrnic (1993).

It is noteworthy that, without σ_α^2 Eq. (42) has the same form as Eq. (18). But, the spectrum width associated with each mechanism in Eq. (42) is weighted by the effective beam pattern. Both σ_α and $H_{en}(\vec{r}_0,\vec{r})$ are functions of *α*. If turbulence, hydrometers' oscillation and/or wobbling and the terminal velocity distribution of hydrometers are locally homogeneous, Eq. (42) then reduces to

$$\sigma_v^2(\vec{r}_0)=\int_V H_{en}(\vec{r}_0,\vec{r})\left[\sigma_o^2(\vec{r})+\sigma_o^2(\vec{r})+\sigma_d^2(\vec{r})+\sigma_s^2(\vec{r})+\sigma_t^2(\vec{r})\right]dV \qquad (45).$$
$$=\sigma_\alpha^2(\vec{r})+\sigma_o^2(\vec{r})+\sigma_d^2(\vec{r})+\overline{\sigma_s^2(\vec{r})}+\sigma_t^2(\vec{r})$$

If we further assume that no shear exists in azimuth direction, Eq. (45) then reduces to the Eq. (5.67) given by Doviak and Zrinc (1993). But, if there is significant shear in the azimuth direction, $\overline{\sigma_s^2(\vec{r})}$ will depend on both the scanning rate α and the azimuth shear k_φ; that is, even without other mechanisms, we cannot separate the second central moment $\sigma_{v\alpha s}^2(\vec{r}_0)=\sigma_\alpha^2(\vec{r})+\overline{\sigma_s^2(\vec{r})}$, into the sum of moments whereby each is associated separately with scan rate α and shear as in Eq. (5.67) of Doviak and Zrnic (1993). Furthermore the conditions for that equation being valid for a scanning radar are: 1. the backscatter sections of hydrometers are independent of turbulence; 2. the influence of both steady and turbulent flows on the weighting function $F_k(mT_s,\vec{r}_0)$ is negligible; 3. oscillation/wobbling of hydrometer is locally homogeneous; 4. turbulence is locally homogeneous; 5. the terminal velocity distribution of hydrometers is locally homogeneous; 6. there is no shear in azimuth direction.

It should be pointed out that we do not have the correlation coefficients with different lagtime in practice. With the PPP algorithm, WSR-88D only uses the correlation coefficient at lag time one. The squared radar observed spectrum width is the estimate of the second moment of the Doppler spectrum (i.e. the Fourier transform of Eq. (38)) under the condition that the Doppler spectrum is Gaussian distribution.

5. The Effect of Effective Beam Width on Spectrum Width

It has been shown in section 4 that, for a scanning radar, if there exists azimuth shear then $\overline{\sigma_s^2(\vec{r})}$ will depend on both the scanning rate α and shear k_φ. We cannot separate the contributions of shear and radar rotation from each other. $\overline{\sigma_s^2(\vec{r})}$ is weighted by $H_{en}(\vec{r}_0,\vec{r})$ which is related to the effective beam pattern. For WSR-88D, the effective beam width is about 1.4 times thestationary beam. Assuming a uniform reflectivity field and $\alpha MT_s \le \theta_1$, $\overline{\sigma_s^2(\vec{r})}$ can be expressed as

$$\overline{\sigma_s^2(\vec{r})}=\int_V I_{en}(\vec{r}_0,\vec{r})\left[v_s(\vec{r})-\bar{v}_s(\vec{r}_p)\right]^2 dV \qquad (46)$$

where $\vec{r}_p=(r_0,\theta_0,\phi_p)$ and I_{en} denotes the effective beam pattern function. It is

$$I_{en}(\vec{r}_0,\vec{r})=\frac{\dfrac{C|W_s(r,r_0)|^2}{l^2(\vec{r})r^4}f_e^4(\phi-\phi_0')f^4(\theta)}{\displaystyle\int_V\frac{C|W_s(r,r_0)|^2}{l^2(\vec{r})r^4}f_e^4(\phi-\phi_0)f^4(\theta)dV} \qquad (47)$$

where

$$f_e^2(\phi-\phi_0')=e^{-\frac{(\phi-\phi_p)^2}{4\sigma_{\phi e}^2}} \qquad (48)$$

$$f_e^2(\phi-\phi_0)=e^{-\frac{(\phi-\phi_p)^2}{4\sigma_{\phi e}^2}} \qquad (49)$$

$$f^2(\theta)=e^{-\frac{(\theta-\theta_0)^2}{4\sigma_\theta^2}} \qquad (50)$$

$$W(r,r_0)=e^{-\frac{(r-r_0)^2}{4\sigma_r^2}} \qquad (51).$$

The analytical expression of $\sigma_{\phi e}^2$ is given by $\sigma_{\phi e}^2=\dfrac{\theta_{1e}^{\ 2}}{16\ln 2\cos^2\theta_e}$ where θ_{1e} is the one way 3 dB beam width of the effective beam. If wind field is linear along azimuth direction and restricted in this direction, it can be shown that

$$\overline{\sigma_{s\phi}^2(\vec{r}_0)} \approx k_\phi^2 r_0^2 \cos^2\theta_e \sigma_{\phi e}{}^2 \qquad (52).$$

For the stationary antenna, from Eq. (5.74) given by Doviak and Zrnic (1993), the spectrum width associated with a linear wind field in azimuth direction is $k_\varphi r_0 \cos\theta_e \sigma_\varphi$. So, the ratio of spectrum width between rotating and stationary antenna due to azimuth shear is

$$\frac{\sqrt{k_\phi^2 r_0^2 \cos^2\theta_e \sigma_{\phi e}^2}}{\sqrt{k_\phi^2 r_0^2 \cos^2\theta_e \sigma_\phi^2}} = \frac{\theta_{1e}}{\theta_1} \qquad (121).$$

Because the effective beam width is usually larger than the one of the stationary beam, this result indicates that if azimuth shear is significant and radar antenna is scanning, the spectrum width due to this shear, i.e. $\overline{\sigma_{s\phi}^2(\vec{r})}$, obtained using stationary beam pattern will underestimate the true value. In this case the effective beam pattern and Eq. (121) should be used.

6. Summery and Conclusions

The Fourier transform of the autocorrelation coefficient of weather radar signal is a weighted normalized expected Doppler spectrum. By decomposing the radial velocity into steady, turbulent and the drop's terminal velocities, all contributors are put in one single unified analytical theoretical frame. Without spectral broadening mechanisms other than shear and turbulence, the weighted normalized expected Doppler spectrum is equal to the probability density function of the total velocity weighted by the beam pattern, range weight and reflectivity. It has been shown that, for a stationary antenna radar, the square of the measured spectrum width is a weighted sum of squared spectrum width associated with each mechanism and two new terms that have heretofore been neglected in the literature. One is the power weighted integral of the spatial gradient of the mean terminal velocity across V_6; another is a cross term depending on the cross product of the gradient of mean terminal velocity of hydrometers and wind. For a scanning radar, the square of the measured spectrum width is still a weighted sum of squared spectrum width associated with each mechanism (except the one associated with antenna rotation) and two new terms, but the effective beam pattern should be applied to replace the regular pattern in this case. If the spectrum related to the terminal velocity is homogeneous, or the reflectivity is uniform across the V_6, the two new terms will disappear. The classic expression, the un-weighted sum of each mechanism, i.e. Eq. 5.67 given by Doviak and Zrnic (1993), is only valid under conditions: 1. the backscatter sections of hydrometers are independent of turbulence; 2. the influence of both steady and turbulent flows on the weighting function is negligible; 3. oscillation/wobbling of the hydrometer is locally homogeneous; 4. turbulence is locally homogeneous; 5. the Doppler spectrum associated with the terminal velocity of hydrometers or the reflectivity is locally homogeneous; 6. there is no shear in azimuth direction. If there exists shear in azimuth direction, the corresponding spectrum width for a scanning radar is larger than that for a stationary antenna radar. Using the formula associated with a stationary antenna will underestimate the contribution of the azimuth shear. The first central moment of the weighted normalized expected normalized Doppler spectrum is contributed by steady flow and mean terminal velocity, and does not contain fluctuations associated with turbulence. So, it is impossible to derive the spatial spectrum of turbulence on all scales from the mean velocity obtained from the expected spectrum. For homogeneous turbulence, the second central moment of the weighted normalized expected Doppler spectrum equals the variance of the turbulent velocity at a point. It contains the contributions from all scales of eddies without any attenuation. Energy partition theory cannot be derived from the expected Doppler spectrum.

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