1 Introduction

Several requirements must be met before the credibility of simulations performed with an atmospheric mesoscale model can be established [1]. Among these requirements we have the comparison with known analytic solutions of the equations of motion. In this work we describe a scheme that yields exact solutions of the bidimensional deep continuity equation [1]

\[ \nabla \cdot \rho_0(r) U(r) = 0 \]  

that satisfy the non-flow boundary condition at an arbitrary representation of terrain \( h(x) \),

\[ U \cdot n = 0 \quad \text{on} \quad z = h(x) \]  

where \( n \) is a vector normal to the lower surface of \( z = h(x) \). The exact solutions of the problem (1,2) are useful to study the reliability of a wide class of methods used to estimate a wind field from the data provided by a monitoring network, namely, the so-called mass consistent models [2]. The exact solutions are obtained by means of a suitable modification of the methods provided by complex variable theory (see, e.g., [3,4]).

The primary problem to obtain the velocity field \( U(r) \) is the solution of the so-called shallow continuity equation [1]

\[ \nabla \cdot V(r) = 0 \]  

under the boundary condition

\[ V \cdot n = 0 \quad \text{on} \quad z = h(x) \]  

In fact, let us suppose that \( V \) is known, then it satisfies

\[ \nabla \cdot \rho_0(r) \frac{V(r)}{\rho_0(r)} = 0 \]  

and the boundary condition

\[ \frac{V(r)}{\rho_0(r)} \cdot n = 0 \quad \text{on} \quad z = h(x). \]

Hence the field

\[ U = \frac{V(r)}{\rho_0(r)} \]

is a solution of equation (1) that satisfies the condition (2). By completeness, in sections 2 and 3 we expose the formal construction of \( V \). Of course, other authors [5] have used complex variable theory to obtain exact solutions of the problem (3,4) but they only consider a computational region with a simple geometry because inherent difficulties of the map-conforming do not permit to solve (3,4) for an arbitrary topography \( h(x) \). In section 4 we illustrate the problems posed by the map conforming to obtain solutions of (3,4). These problems can be circumvented if the topography \( h(x) \) is estimated by means of cubic splines [6], which are defined in section 4. In section 5 we describe the field \( V(r) \) obtained from the natural spline corresponding to real terrain elevation data from the data-base GTOPO30 [7]. It is shown that the map-conforming and the splines generate an exact field \( V \) even when the terrain elevation \( h(x) \) changes suddenly as \( x \) increases. Histograms of the components of \( V = u_i + u_k \) show that \( u \) and \( w \) have a very irregular behavior. Similar results are obtained with the components of the field \( U = \rho_0^{-1} V \) where \( \rho_0 \) is the density corresponding to an atmospheric reference state that is isothermic or adiabatic. The field \( U \) is more interesting because its vorticity is not trivial,

\[ \nabla \times U \neq 0 . \]
a property that is exhibited by the behavior of the components of $\mathbf{U}$ as a function of the height $z$.

There is no consensus about the correct representation of the topography in atmospheric models. In general, the terrain data are subject to a smoothing process. Cubic splines can be used to smooth terrain elevation data and, consequently, we can compute the fields $\mathbf{V}$ and $\mathbf{U}$. In section 5 we describe the smoothing splines and the corresponding fields $\mathbf{V}$ and $\mathbf{U}$. The results show that the $\mathbf{V}$ and $\mathbf{U}$ are critically dependent of $h(x)$.

\section{Formal construction of the field $\mathbf{V}$}

In this section we give a summary of the theory of bidimensional potential flow. Let $x$, $y$, $z$ be the a cartesian coordinate system with its origin on a spherical earth model, the $z$ axis out of the sphere and the plane $xy$ is tangent to the sphere at a point with latitude $\phi$ and longitude $\lambda$, the corresponding unit vectors are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. If the field $\mathbf{V}$ is irrotational,

$$\nabla \times \mathbf{V} = 0$$

there exists a function $\phi$ (the velocity potential) such that

$$\mathbf{V} = \nabla \phi.$$  \hspace{1cm} (5)

If $\mathbf{V}$ satisfies (3) and (4), then $\phi$ has to satisfy the Laplace equation

$$\nabla^2 \phi = 0$$  \hspace{1cm} (6)

and the boundary condition

$$\frac{\partial \phi}{\partial n} \equiv \nabla \phi \cdot \mathbf{n} = 0 \quad \text{on} \quad z = h(x).$$  \hspace{1cm} (7)

The velocity field $\mathbf{V}$ can be obtained by solving (6) with the boundary condition (7) without directly using the momentum equation

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

which can be used to obtain the pressure field. A second function can be defined in such a way that the associated velocity field automatically satisfies the equation (3). In fact, the field $\mathbf{V} = \mathbf{i}u + \mathbf{k}w$ with components

$$u = \frac{\partial \psi}{\partial z} \quad w = -\frac{\partial \psi}{\partial x}$$  \hspace{1cm} (9)

satisfies (3) for any function $\psi$ (the stream function). In vectorial form $\mathbf{V}$ is given by

$$\mathbf{V} = \mathbf{j} \times \nabla \psi.$$  \hspace{1cm} (10)

The irrotational condition $\nabla \times (\mathbf{j} \times \nabla \psi) = 0$ that $\psi$ must satisfy, takes the form $\nabla^2 \psi = 0$. The function $\psi$ has useful properties that allow us to obtain the field $\mathbf{V}$ that satisfies (3) and (4) with an arbitrary topography $h(x)$. Let $\mathbf{r}(\eta) = x(\eta) \mathbf{i} + z(\eta) \mathbf{k}$ be the equation of a curve where $\psi$ has a constant value $\psi_0$,

$$\psi(x(\eta), z(\eta)) = \psi_0,$$

then

$$\nabla \psi \cdot \frac{d\mathbf{r}(\eta)}{d\eta} = 0$$

From (10) we get $\nabla \psi = -\mathbf{j} \times \mathbf{V}$ and replacing in the last equation we have

$$(-\mathbf{j} \times \mathbf{V}) \cdot \frac{d\mathbf{r}(\eta)}{d\eta} = (\mathbf{V} \times \frac{d\mathbf{r}(\eta)}{d\eta}) \cdot \mathbf{j} = 0.$$  \hspace{1cm} (11)

Since $\mathbf{V}$ and $\mathbf{r}(\eta)$ belong to the $xz-$plane, it follows that $\mathbf{V}$ is parallel to the vector $d\mathbf{r}/d\eta$; that is, $\mathbf{V}$ is tangent to the curve $\mathbf{r}(\eta)$. Thus we have

$$u = \frac{\partial \psi}{\partial z} = \frac{\partial \phi}{\partial x} \quad w = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial z}.$$  \hspace{1cm} (11)

The relationships between the derivatives of $\psi$ and $\phi$ are the so-called Cauchy-Riemann equations, which imply that $\psi$ and $\phi$ are the components of the function (called the complex potential of $\mathbf{V}$)

$$F(\xi) = \phi(x, z) + i\psi(x, z)$$

of the complex variable

$$\xi = x + iz$$

in terms of which $\mathbf{V}$ is given by

$$\mathbf{V} = \mathbf{V} = \frac{dF(\xi)}{d\xi} = u - iw.$$  \hspace{1cm} (12)

An important property of complex analytic functions to obtain our velocity field $\mathbf{V}$ is that they keep the angle between curves. In fact, let us consider an abstract complex plane

$$\zeta = \bar{x} + i\bar{z}$$

and let

$$\bar{F}(\zeta) = \bar{\phi}(\bar{x}, \bar{z}) + i\bar{\psi}(\bar{x}, \bar{z})$$

be a complex potential so that

$$\bar{\mathbf{V}} = \frac{d\bar{F}(\zeta)}{d\zeta} = \bar{u} - i\bar{w}.$$
defines a velocity field in the $\zeta$ plane with potential $\phi = \text{Re} \hat{F}(\zeta)$ and stream function $\psi = \text{Im} \hat{F}(\zeta)$. Let us consider that there is a relationship between $\xi$ and $\zeta$,

$$\xi = G(\zeta),$$

and let us denote the inverse transformation by $G^{-1}$,

$$\zeta = G^{-1}(\xi).$$

The complex potential

$$F(\xi) = \hat{F}[G^{-1}(\xi)] = \phi(x, z) + i\psi(x, z)$$

yields a velocity field $V$ (12) in the $xz$-plane that is the image of $\hat{V}$ under the transformation $G(\zeta)$. Suppose that $G(\zeta)$ is an analytic function of $\zeta$, then $G$ transforms stream lines of the flow $\hat{V}$ into the stream lines of the field $V$ and analogously with the equipotential lines [4]. Thus, if $G$ transforms a stream line of $\hat{V}$ into the curve $z = h(x)$, the flow $V$ automatically satisfies the continuity equation (3) and the boundary condition (4).

In this way, the problem to compute our desired velocity field $V$ consists in defining a field $\hat{V}$ in the $\zeta$-plane with a stream line $\hat{\psi} = \psi_0$ whose image under an analytic function $G(\zeta)$ is the curve $z = h(x)$.

Suppose that we know a field $\hat{V}$ and a function $G(\zeta)$ with the aforementioned properties. The desired field $V$ can be obtained from the stream function $\psi(x, y)$ or the potential $\phi(x, y)$ as follows. In principle, from the inverse transformation $G^{-1}$,

$$\zeta = \bar{x} + i\bar{z} = G^{-1}(\xi) = \bar{x}(x, z) + i\bar{z}(x, z)$$

(13a)

we get the inverse transformation equations

$$\bar{x} = \bar{x}(x, z), \quad \bar{z} = \bar{z}(x, z).$$

(13b)

The substitution of these expressions in the equation

$$F(\xi) = \hat{F}[G^{-1}(\xi)]$$

yields the relationship between $\phi$, $\psi$ and $\bar{\phi}$, $\bar{\psi}$, namely,

$$\phi(x, z) = \bar{\phi}[\bar{x}(x, z), \bar{z}(x, z)]$$

and, therefore, the desired field $V$ has the components

$$u = \frac{\partial \bar{\psi}}{\partial z} = \frac{\partial \bar{\phi}}{\partial x},$$

$$w = -\frac{\partial \bar{\psi}}{\partial x} = \frac{\partial \bar{\phi}}{\partial z}.$$
3 The case of a uniform flow $\vec{V}$ in the plane $\zeta$

In this section we consider the simplest flow $\vec{V}$ in the complex plane $\zeta$ and an analytic function $G(\zeta)$ that transforms a stream line of $\vec{V}$ into a terrain curve $z = h(x)$. The flow in question is the uniform field

$$\vec{V} = V_0 \quad (u = V_0, \ w = 0)$$

obtained from the potential

$$\phi = V_0 \bar{x}.$$ 

The corresponding flow $V$ under an arbitrary analytic transformation $G(\zeta)$ is [eq. (16)]

$$u = \frac{V_0}{T} \frac{\partial z}{\partial \bar{z}}$$

$$w = \frac{V_0}{T} \frac{\partial x}{\partial \bar{z}}$$

The real axis $\bar{z} = 0$ in the $\zeta$-plane is a stream line of $\bar{V}$, and the simplest function $G(\zeta)$ that transforms such an axis into the curve $z = h(x)$ is

$$G(\zeta) = \zeta + ih(\bar{x}).$$

In fact, if $\zeta$ is replaced by $\zeta = \bar{x}$ we get

$$G(\zeta) = \bar{x} + ih(\bar{x})$$

which is exactly the parametric form of the curve $z = h(x)$ with $x = \bar{x}$. Let $h_1$ and $h_2$ be the real and imaginary parts of $h(\zeta)$,

$$h(\zeta) = h_1(\bar{x}, \ \bar{z}) + ih_2(\bar{x}, \ \bar{z}),$$

then the transformation equations defined by (15) are

$$x = x(\bar{x}, \ \bar{z}) = \bar{x} + h_2(\bar{x}, \ \bar{z})$$

$$z = z(\bar{x}, \ \bar{z}) = \bar{z} + h_1(\bar{x}, \ \bar{z}).$$

4 Problems posed by conforming map and splines

In principle any (analytic) representation of the terrain $h(\zeta)$ can be used in the transformation $G(\zeta) = \zeta + ih(\zeta)$. For instance, let us consider the topography

$$h(\zeta) = \sin \pi \zeta/a$$

where $a$ is a positive real number. The corresponding transformation equations (19) are

$$x = x(\bar{x}, \ \bar{z}) = \bar{x} - \cos(\pi \bar{x}/a) \ \sinh(\pi \bar{z}/a)$$

$$z = z(\bar{x}, \ \bar{z}) = \bar{z} + \sin(\pi \bar{x}/a) \ \cosh(\pi \bar{z}/a).$$

The presence of the hyperbolic functions implies that a region in the $\zeta$-plane is substantially different in the $\bar{x}$-plane. Figure 1 shows that the image of the semiplane $\{ \zeta : \text{Im} \zeta \geq 0 \}$ under the transformation $\zeta = \sin \pi \zeta/a$, is a very small region of the $\bar{x}$-plane (the physical space). This implies that we cannot compute the desired field $V$ on an arbitrary region of the physical space. This example also shows the convenience of computing the field $V$ using $\bar{x}$ and $\bar{z}$ as independent variables [eq. (16)], since in general the inverse transformation (13a,b) cannot be obtained in a closed and analytic form because of the non-linearity of the direct transformation (19) as occurs with (20).

Figure 1: Region in the $\zeta$-plane and its image in the $\bar{x}$-plane with $h(\zeta) = \sin \pi \zeta/a$.

Since our primary objective is the calculation of a velocity field $V$ that satisfies (3) and the boundary condition (4) we can replace an arbitrary (but analytic) topography $h(z)$ by a simpler representation that eliminates the inherent problem of the map conforming. In this section $h(x)$ is approximated by a natural spline $S(x)$ which is defined as follows. Let $\{x_k\}_{k=0}^n$ be a set of points where the terrain height $h(x_k)$ is known, then: (i) $S(x)$ satisfies

$$S(x_k) = h(x_k) \quad \text{for} \quad k = 0, \ldots, n,$$

(ii) $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$,

$$S(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3 \quad \text{for} \quad x \in [x_k, x_{k+1}],$$

(iii) $S(x)$ and its derivatives $S'(x)$, $S''(x)$ are continuous on $[x_0, x_n]$ and $S''(x)$ satisfies

$$S''(x_0) = S''(x_n) = 0.$$
There is a unique natural spline associated to an analytic function $h(x)$ on the interval $[x_0, x_n]$. Since $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$, we can compute the flow

$$
u = \frac{V_0}{J} \frac{\partial \tilde{z}^{(k)}}{\partial \tilde{x}},$$
$$w = -\frac{V_0}{J} \frac{\partial \tilde{x}^{(k)}}{\partial \tilde{z}}$$

for $x \in [x_k, x_{k+1}]$, where

$$\tilde{x}^{(k)} = \tilde{x} - S_2(\tilde{x}, \tilde{z})$$
$$\tilde{z}^{(k)} = \tilde{z} + S_1(\tilde{x}, \tilde{z})$$

and

$$S(\zeta = \tilde{x} + i \tilde{z}) = S_1(\tilde{x}, \tilde{z}) + i S_2(\tilde{x}, \tilde{z}).$$

The continuity of $S(x)$, $S'(x)$ and $S''(x)$ guarantees that the field $\mathbf{V} = u \mathbf{i} + w \mathbf{k}$, its first derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial z}$$

and $\nabla \cdot \mathbf{V}$ are continuous on the interval $[x_0, x_n]$. This together with the fact that $u$, $w$ satisfy the continuity equation (3) and the boundary condition (4) on each interval $[x_k, x_{k+1}]$, implies that the field $\mathbf{V}$ satisfies the same equations on the whole interval $[x_0, x_n]$.

**Remark.** A clear advantage of the splines is that they can be used to model real terrain elevation data $h(x_k)$ which are known only on a discrete set of points $\{x_k, h(x_k)\}$ which constitute a digital terrain elevation model.

5 Examples

The velocity fields considered below are calculated with the datum $u = 10 \text{ ms}^{-1}$ and $w = 0$ at the point $(x = 0, z = 10 \text{ km})$, which is used to define the magnitude $V_0$ of the flow on the abstract complex plane $\zeta$.

5.1 Flow $\mathbf{V}$ from a natural spline

In this section we consider the field $\mathbf{V}$ defined by the natural spline corresponding to the terrain data $\{x_k, h(x_k)\}$ of fig. 2 with $-400 \leq x \leq 400 \text{ km}$, the data were obtained from the base GTOPO30 [7]. The field $\mathbf{V}$ is a solution of the shallow continuity equation (3) and the boundary condition (4), with $\rho$ constant. Hereafter, we describe the part of the field on the region $-10 \leq x \leq 10 \text{ km}$. Figures 3 and 4 show $\mathbf{V}(x, z)$ on lines with $x = x_0 = \text{constant}$. 

Figure 2: Topography used to compute a natural spline $S(x)$.

![Figure 2](image2)

Figure 3: Field $\mathbf{V}(x_0, z)$ at points with $x_0 = \text{constant}$.

Figure 4: Field $\mathbf{V}(x, z_0)$ at points with $z_0 = \text{constant}$.
and $z = z_0 = \text{constant}$, respectively. Each figure shows the effect of the topography on $V(x, z)$.

Figures 5 and 6 show the plot of $z \text{ v.s. } u(x_0, z)$ and $w(x_0, z)$, respectively, at $x_0 = -10, 0, 10$ km. The graphs allows us to see the continuity of $u(x, z)$ and $w(x, z)$ as $z$ increases from the topography $z = h(x)$.

Figures 7 and 8 show the plot of $x \text{ v.s. } u(x, z_0)$ and $w(x, z_0)$, respectively, at $z_0 = 2, 10$ km. The graphs show that $u(x, z)$ and $w(x, z)$ have an irregular behavior as $x$ increases, a result that can be attributed to the topography. The case of $w(x, z_0 = 2)$ is particularly interesting since we observe that it behaves very irregularly. In principle, the continuity of the spline and its first two derivatives should produce smooth graphs of $u(x, z_0)$ and $w(x, z_0)$, but the figures 7 and 8 allows us to see that the field $V(x, z)$ keeps the sudden changes of the topography.
In order to show that the continuity equation $\nabla \cdot V(x, z) = 0$ is satisfied we plot the values $\nabla \cdot V(x, z)$ at points $(x_0, z)$ and $(x, z_0)$ in figures 9 and 10, respectively. As expected, we see that $\nabla \cdot V(x, z)$ is, for practical purposes, zero.

Let us now consider the boundary condition $V \cdot n = 0$ on $z = h(x)$. As expected, figure 11 shows that $V \cdot n$ is essentially the zero of the computer machine. As above, the irregular behavior is due to the irregularity of the topography.

In order to appreciate the irregular behavior of the components $u(x, z_0)$ and $w(x, z_0)$ as a function of $x$ with $-10 \leq x \leq 10$, we plotted some histograms. The histogram of $w(x, z_0)$ with $z_0 = 2$ km is plotted in figure 12, and we see that it behaves like a Gaussian distribution with $< w > \sim 0$. Figure 13 shows the histogram of $w(x, z_0 = 10)$ km, and we see that it behaves like a three-modal distribution. Finally, figures 14 and 15 show the histogram of $u(x, z_0)$ at $z_0 = 2$ and $10$ km, respectively. The behavior of $u$ at $z_0 = 2$ km is irregular but with a small dispersion, in contrast the behavior at $z_0 = 10$ km is significantly more irregular on a wider range of velocities.

### 5.2 Flow $V$ from a smoothing spline

In this section we consider the field $V$ defined by a smoothing spline $S_p(x)$ corresponding to the terrain data $\{x_k, h(x_k)\}_{k=1}^N$ of fig. 2. The field $V$ is a solution of the shallow continuity equation (3) and the boundary condition (4), $\rho = $ constant, and we describe the part of the field on the region $-10 \leq x \leq 10$ km. The splines $S_p(x)$
are obtained by minimizing the functional

\[ F = p \sum_{k=1}^{n} |h_k - S_p(x_k)|^2 + (1 - p) \int_{x_1}^{x_n} |D^2 S_p(t)|^2 dt \]

Here, \( n \) is the number of terrain data and the integral is over the smallest interval containing all the data \( x_k \). Further, \( D^2 S_p(t) \) denotes the second derivative of \( S_p(x) \). The smoothing parameter is \( p \), it determines the relative weight on the contradictory demands of having be smooth v.s. having be close to the data. For \( p = 0 \), we get the least-squares straight line fit to the data, while, at the other extreme, \( p = 1 \) yields the natural spline. As \( p \) moves from 0 to 1, the smoothing spline changes from one extreme to the other.

We begin with the field \( \mathbf{V}_{p=0.1}(x,z) \) from the smoother spline \( S_p(t) \) with \( p = 0.1 \). Figures 16 and 17 show \( \mathbf{V}_{p=0.1}(x,z) \) on lines with \( x = x_0 \) constant and \( z = z_0 \) constant, respectively. Figures 18 and 19 show the plot of \( z \) v.s. \( u(x_0, z) \) and \( w(x_0, z) \), respectively, at \( x_0 = -10, 0, 10 \) km, where we see the continuity of \( u(x,z) \) and \( w(x,z) \) as \( z \) increases from the topography \( z = h(x) \). Figures 20, 21 show the plot of \( x \) v.s. \( u(x_0,z_0) \) and \( w(x_0,z_0) \), respectively, at \( z_0 = 2, 10 \) km. As expected, \( u \) and \( w \) are smooth at \( z_0 = 2 \) km but they become irregular as \( z_0 \) increases. The irregularity can be seen in the histograms plotted in figs. 22 to 25.

Figures 26 to 31 show the behavior of \( \mathbf{V}_{p=0.5}(x,z) \) from the smoothing spline \( S_p(t) \) with \( p = 0.5 \). Figures 26 and 27 show \( \mathbf{V}_{p=0.5}(x,z) \) on lines with \( x = x_0 \) and \( z = z_0 \), respectively. Figures 28 and 29 show the plot of \( z \) v.s. \( u(x_0, z) \) and \( w(x_0, z) \), respectively, at \( x_0 = -10, 0, 10 \) km.

Figure 13: Histogram of \( w(x, z_0) \) for \(-10 \leq x \leq 10 \) and \( z_0 = 10 \) km.

Figure 14: Histogram of \( u(x, z_0) \) for \(-10 \leq x \leq 10 \) and \( z_0 = 2 \) km.

Figure 15: Histogram of \( u(x, z_0) \) for \(-10 \leq x \leq 10 \) and \( z_0 = 2 \) km.
Figure 16: Field $V_{p=0.1}(x_0, z)$ at points with $x_0 =$ constant.

Figure 17: Field $V_{p=0.1}(x, z_0)$ at points with $z_0 =$ constant.

Figure 18: Graph of $z$ v.s. $u_{p=0.1}(x_0, z)$ at $x_0 = -10, 0, 10$ km.

Figure 19: Graph of $z$ v.s. $w_{p=0.1}(x_0, z)$ at $x_0 = -10, 0, 10$ km.

Figure 20: Graph of $x$ v.s. $w_{p=0.1}(x, z_0)$ at $z_0 = 2, 10$ km.
Figure 21: Graph of $x$ v.s. $w_{p=0.1}(x,z_0)$ at $z_0 = 2, 10$ km.

Figure 22: Histogram of $w_{p=0.1}(x,z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 2$ km.

Figure 23: Histogram of $w_{p=0.1}(x,z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 10$ km.

Figure 24: Histogram of $u_{p=0.1}(x,z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 2$ km.

Figure 25: Histogram of $u_{p=0.1}(x,z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 2$ km.
km, where we see the continuity of $u(x, z)$ and $w(x, z)$ as $z$ increases from $z = h(x)$. Figures 30, 31 show the plot of $x$ v.s. $u(x, z_0)$ and $w(x, z_0)$, respectively, at $z_0 = 2, 10$ km. Once again, $u$ and $w$ are smooth at $z_0 = 2$ km but they become irregular as $z_0$ increases.

Figures 26 to 39 show the behavior of $\mathbf{V}_{p=0.9}(x, z)$ from the smoothing spline $S_p(t)$ with $p = 0.9$. Figures 32 and 33 show $\mathbf{V}_{p=0.9}(x, z)$ on lines with $x = x_0$ and $z = z_0$, respectively. Figures 34 and 35 show the plot of $z$ v.s. $u(x_0, z)$ and $w(x_0, z)$, respectively, at $x_0 = -10, 0, 10$ km, where we see the continuity of $u(x, z)$ and $w(x, z)$ as $z$ increases from $z = h(x)$. The irregularity can be seen in the histograms plotted in figs. 36 to 39.
Figure 30: Graph of $x$ v.s. $w_{p=0.5}(x, z_0)$ at $z_0 = 2, 10$ km.

Figure 31: Graph of $x$ v.s. $w_{p=0.5}(x, z_0)$ at $z_0 = 2, 10$ km.

Figure 32: Field $V_{p=0.9}(x_0, z)$ at points with $x_0 =$ constant.

Figure 33: Field $V_{p=0.9}(x, z_0)$ at points with $z_0 =$ constant.

Figure 34: FIG 34 Graph of $z$ v.s. $u_{p=0.9}(x_0, z)$ at $x_0 = -10, 0, 10$ km.

Figure 35: Graph of $z$ v.s. $w_{p=0.9}(x_0, z)$ at $x_0 = -10, 0, 10$ km.
Figure 36: Histogram of $w_{p=0.9}(x, z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 2$ km.

Figure 37: Histogram of $w_{p=0.9}(x, z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 10$ km.

Figure 38: Histogram of $u_{p=0.9}(x, z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 2$ km.

Figure 39: Histogram of $u_{p=0.9}(x, z_0)$ for $-10 \leq x \leq 10$ and $z_0 = 10$ km.

Figure 40 shows the fields $V_p$ with $p = 0.1, 0.5, 0.9, 1.0$ where $p = 1$ corresponds to the natural spline. A comparison of the above results shows that "small" changes on the topography (approximated by a spline) produce significant changes in the field $V_{p=0.5}$. This is particularly evident from the histograms. The most irregular field is $V_{p=1.0}$. In principle we can expect that $V_p$ tends to $V_{p=1.0}$ as $p \to 1.0^-$ but this limiting process is not continuous since the natural spline $S_{p=1}(x)$ is almost discontinuous (see fig. 2).

Figure 40: Splines $S_p(x)$ and the corresponding velocity fields $V_p$ for $p = 0.1, 0.5, 0.9, 1.0$.

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6 References


