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RELIABILITY ANALYSIS OF MASS CONSISTENT MODELS

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1 Introduction

Mass Consistent Models (MCM's) [1-8] are some of the main assimilation schemes of wind data by the simplicity of the physics involved and their capacity to accept wind data from different points in a region. Additionally, these provide a way to estimate initial meteorological fields as input of prognostic models [1,9,10]. The MCM's are based on the minimization of the variance between an initial field $\mathbf{V}^0$ (obtained from a suitable interpolation of discrete data) and a field $\mathbf{V}$ (called the adjusted field) that satisfies the mass conservation equation $\nabla \cdot \mathbf{V} = 0$. The variance is usually defined in terms of a set of parameters α_i. The scheme leads to an elliptic problem for a Lagrange multiplier λ. A careful deduction of the minimization process is given in section 2 [11]. Two kind of boundary conditions on the topography, have been used in the literature to compute λ [1]: (i) The Nuemann boundary condition $\partial\lambda/\partial n = 0$ and (ii) the no-flow condition $\mathbf{V} \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is normal to the topography. It is shown that the former is incorrect when $\alpha_i \neq 1$. All simulations performed with MCM's show that the adjusted field $\mathbf{V}$ is very sensitive to the values of the α_i's but there is no concensus about its correct value. For instance, some authors suggest that the α_i's should be estimated by considering the residual divergence [4]. The deduction of section 2 shows that the exact λ yields a zero residual divergence for any choice of the α_i's, so that sensibility of the residual divergence to the α_i's is a consequence of the numerical estimation of λ rather than an intrinsic property of the scheme.

Some authors [8] have studied the reliability of MCM's by comparison of the adjusted field $\mathbf{V}$ with a true field $\mathbf{V}_{\rm true}$ obtained from Map conforming. The analysis is partial because (i) the adjusted field is calculated with $\alpha_i = 1$ and (ii) the map conforming provides analytic velocity fields only on a region with a simple geometry, since inherent problems of the map conforming do not permit consider a region with a complex topography. These problems are solved in two-dimensions when the topography is approximated by means of splines [12] and the resulting flow is used in this work to study the reliability of MCM's (section 3). The finite element method [13] is used to estimate λ (section 4) and the numerical results are reported in section 5. We consider the flow on a smooth topography with data from the data base GTOPO30 [14]. The results show the adjusted field $\mathbf{V}$ can reproduce the main characteristics of the true flow $\mathbf{V}_{\rm true}$ with a suitable choice of the α_i's.

2 Theoretical formulation of MCM's

Let us consider a cartesian coordinate system xyz with its origin at a point on a spherical earth model, the xy plane is tangent to the earth and the z axis is out of the earth. Hereafter we employ the notation $x^1 = x$, $x^2 = y$, $x^3 = z$ and $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1$, $\hat{\mathbf{y}} = \hat{\mathbf{x}}_2$, $\hat{\mathbf{z}} = \hat{\mathbf{x}}_3$ to denote the unit vectors associated to the system xyz. The region of study is $D = \{x_{\min} \leqslant x \leqslant x_{\max},\ y_{\min} \leqslant y \leqslant y_{\max},\ h(x,y) \leqslant z \leqslant z_{\max}\}$ where $h(x,y)$ is the terrain elevation

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on $(x, y, z = 0)$. The true velocity field is denoted by

$$\mathbf{V}_{\text{true}} = \sum_i V^i_{\text{true}} \hat{\mathbf{x}}_i \ .$$

By simplicity we consider that $\mathbf{V}_{\text{real}}$ satisfies the shallow continuity equation

$$\nabla \cdot \mathbf{V}_{\text{true}} = 0 \ , \qquad \left(\nabla \equiv \sum \hat{\mathbf{x}}_i \frac{\partial}{\partial x^i} \right),$$

and the lower boundary condition

$$\mathbf{V}_{\text{true}} \cdot \mathbf{n} = 0 \quad \text{for} \quad z = h(x, y)$$

where $\mathbf{n}$ is a vector normal to the topography,

$$\mathbf{n} = \nabla(h - z) = \hat{\mathbf{x}} \partial_x h + \hat{\mathbf{y}} \partial_y h - \hat{\mathbf{z}} \ .$$

Let us suppose that the data from a meteorological network are interpolated in such a way that they provide and initial field

$$\mathbf{V}^0 = \sum_i V^{i0} \ \hat{\mathbf{x}}_i \, ,$$

In general , the field $\mathbf{V}^0$ does not satisfy the equation $\nabla \cdot \mathbf{V}^0 = 0$, so that the problem consists in estimating an adjusted field $\mathbf{V}$ that satisfies $\nabla \cdot \mathbf{V} = 0$, the boundary condition $\mathbf{V} \cdot \mathbf{n} = 0$ on $z = h(x, y)$ and is closer to $\mathbf{V}^0$ with respect to some suitable distance $d(\mathbf{W}, \mathbf{U})$ between vector fields.

2.1 A general definition of the distance $d(\mathbf{W}, \mathbf{U})$

In order to define a metric $d(\mathbf{W}, \mathbf{U})$ we use the following notation. The standard Euclidean inner product is denoted by

$$\mathbf{W} \cdot \mathbf{U} = \sum_i W^i \hat{\mathbf{x}}_i \cdot \sum_j U^j \hat{\mathbf{x}}_j = \sum_i W^i U^i \, ,$$

We assume that $\mathbf{W}$ denotes the column vector

$$\mathbf{W} = \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix}$$

and, consequently, the transpose$\mathbf{W}^t$ denotes a row vector,

$$\mathbf{W}^t = \begin{pmatrix} W^1 & W^2 & W^3 \end{pmatrix} \ .$$

According to this notation and the usual matrix algebra the inner product $\mathbf{W} \cdot \mathbf{U}$ can be written as $\mathbf{W}^t \mathbf{U}$. Let $\mathbb{S}$ be a symmetric and positive definite matrix, then the expression

$$\langle \mathbf{W} \mid \mathbf{U} \rangle_S = \mathbf{W}^t \ \mathbb{S} \ \mathbf{U}$$

defines a general inner product.

The vector fields of interest depend of each point $x \equiv (x^1, x^2, x^3)$,

$$\mathbf{W} = \mathbf{W}(x) = \sum_i W^i(x) \ \hat{\mathbf{x}}_i.$$

In this case we combine the inner product in function spaces

$$< f | g >= \int_D f(x) g(x) dx \ .$$

with $\langle \mathbf{W} \mid \mathbf{U} \rangle_S$ to get to the following inner product for vector fields

$$\langle \mathbf{W} \mid \mathbf{U} \rangle_{DS} \equiv \int_D \langle \mathbf{W} \mid \mathbf{U} \rangle_S \ dx = \int_D \mathbf{W}^t \ \mathbb{S} \ \mathbf{U} \ dx.$$

where we assume that the matrix $\mathbb{S}(x)$ is symmetric and positive definite at each point x in D. The corresponding norm and metric are, respectively,

$$\|\mathbf{W}\|^2_{DS} = \langle \mathbf{W} \mid \mathbf{W} \rangle_{DS} = \int_D \|\mathbf{W}\|^2_S \ dx$$

and

$$d_{DS}(\mathbf{W}, \mathbf{U}) = \|\mathbf{W} - \mathbf{U}\|_{DS} \ .$$

In what follows the boundary of the region D is denoted by ∂D and we consider the following decomposition

$$\partial D = \partial D_a \cup \partial D_b$$

where ∂D_a denotes the lateral and upper boundaries of D and ∂D_b denotes the lower boundary defined by the topography $z = h(x, y)$. The problem of estimating an adjusted field $\mathbf{V}$ from a given initial field $\mathbf{V}^0$ consists in computing the field $\mathbf{V}$ that is closer to $\mathbf{V}^0$ with respect to the metric

$$\left\|\mathbf{V} - \mathbf{V}^0\right\|^2_{DS} = \int_D \left\|\mathbf{V} - \mathbf{V}^0\right\|^2_S \ dx \, ,$$

and satisfies the equation

$$\nabla \cdot \mathbf{V} = 0 \qquad \text{in } D \, ,$$

together with the boundary condition

$$\mathbf{V} \cdot \mathbf{n} = 0 \qquad \text{on } \partial D_b \, .$$

The conditions that the adjusted field $\mathbf{V}$ has to satisfy, imply that it belongs to the set

$$\mathbb{V}= \{\mathbf{W}(x) \in \mathbf{L}_2(D) : \nabla \cdot \mathbf{W} = 0 \text{ in } D \quad \text{and } \mathbf{W} \cdot \mathbf{n} = 0 \text{ on } \partial D_b\} \ ,$$

which constitues a vector space. The distance between a vector field $\mathbf{W}$ in $\mathbb{V}$ and the initial field $\mathbf{V}^0$ defines the functional

$$F_S(\mathbf{W}) \equiv \left\|\mathbf{W} - \mathbf{V}^0\right\|^2_{DS} = \int_D \left\|\mathbf{W} - \mathbf{V}^0\right\|^2_S \ dx \, ,$$

in terms of which the calculation of an adjusted field consists in estimating the field $\mathbf{V}$ in $\mathbb{V}$ that satisfies

$$F_S(\mathbf{V}) = \min_{\mathbf{W}\in\mathbb{V}} F_S(\mathbf{W}) \ .$$

2.2 Contravariant form of $F_S(\mathbf{W})$

A standard procedure to simplify the treatment of the lower boundary condition $\mathbf{W} \cdot \mathbf{n} = 0$ on ∂D_b consists in using the so-called terrain-following coordinates. Since there is no consensus about the definition of these coordinates, in this section we report the expression of the functional $F_S(\mathbf{W})$ in an arbitrary coordinate system $y^1y^2y^3$ which is defined by means of a set of transformation equations

$$y^j = y^j(x) \qquad j = 1, 2, 3 \, , \tag{1}$$

where the time t does not appears explicitly. The equations of the inverse transformation have the form

$$x^i = x^i(y) \qquad i = 1, 2, 3 \ . \tag{2}$$

The $i-$th row and the $j-$th column of jacobian matrix corresponding to (1) are defined by

$$\mathbb{J}_{ij} = \frac{\partial x^i}{\partial y^j} \, .$$

The metric tensor is defined by

$$\mathbb{G} = \mathbb{J}^t\mathbb{J} \, , \tag{3}$$

and its elements are denoted by g_{kl},

$$g_{kl} \equiv \mathbb{G}_{kl} = \sum_i \frac{\partial x^i}{\partial y^k}\frac{\partial x^i}{\partial y^l} \, .$$

The elements of the inverse matrix $\mathbb{G}^{-1}$ are denoted by g^{kl},

$$g^{kl} \equiv (\mathbb{G}^{-1})_{kl} \ .$$

We suppose that the jacobian of the transformation (2) is positive,

$$J(y) = \det(\mathbb{J}) > 0 \, \text{on } D.$$

The determinant of $\mathbb{G}$ is usually denoted by g and from (3) we get the relationship

$$g = \det(\mathbb{J}^t\mathbb{J}) = J^2 \qquad \text{or} \qquad \sqrt{g} = J \ .$$

If $J(y)$ is not zero on D, the so-called covariant vectors

$$\tau_j = \sum_i \frac{\partial x^i}{\partial y^j}\hat{\mathbf{x}}_i \qquad j = 1, 2, 3 \, , \tag{4}$$

are linearly independent and, therefore, constitute a basis set of $\mathbb{R}^3$. In an analogous manner the contravariant vectors

$$\eta^j \equiv \nabla y^j(x) = \sum_i \frac{\partial y^j}{\partial x^i}\hat{\mathbf{x}}^i \tag{5}$$

constitute a base of $\mathbb{R}^3$. The relationship between τ_j y η^k is

$$\tau_j = \sum_k g_{jk}\eta^k \qquad \eta^k = \sum_j g^{kj}\tau_j \ . \tag{6}$$

and using the chain rule it is easy to see that τ_j and η^k satisfy the reciprocity relationship

$$\tau_j \cdot \eta^k = \delta^k_j. \tag{7}$$

Since the basis $\hat{\mathbf{x}}_i$ associated to the cartesian coordinates is orthonormal ($\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij}$) we can write

$$\hat{\mathbf{x}}_i = \hat{\mathbf{x}}^i \ .$$

The expression of a vector $\mathbf{U}$ in the basis τ_j

$$\mathbf{U} = \sum_j U^j(y) \ \tau_j \tag{8}$$

is called the contravariant form of $\mathbf{U}$, and the coefficients $U^j(y)$ are the contravariant components of $\mathbf{U}$ in the coordinate system y. The expression of $\mathbf{U}$ in the basis η^j

$$\mathbf{U} = \sum_j U_j(y) \ \eta^j \tag{9}$$

is known as the covariant form of $\mathbf{U}$ and the coefficients $U_j(y)$ are the covariant components of $\mathbf{U}$ in the coordinate system y. Accordingly, the contravariant and covariant forms of $\mathbf{U}$ in the cartesian coordinate system x coincide,

$$\mathbf{U} = \sum_i U^i(x) \ \hat{\mathbf{x}}_i = \sum_i U_i(x) \ \hat{\mathbf{x}}^i$$

where we have $U^i(x) = U_i(x)$.

From (4) and (8) we get the rationship

$$U^j(y) = \sum_i \frac{\partial y^j}{\partial x^i} U^i(x) \tag{10}$$

which is known as the contravariant transformation law. From (5) en (9) we get the covariant law

$$U_j(y) = \sum_i \frac{\partial x^i}{\partial y^j} U_i(x) \ . \tag{11}$$

Finally, from (6) we get the relationship between $U^j(y)$ and $U_j(y)$

$$\begin{aligned} U_j(y) &= \sum_k g_{jk} U^k(y) \\ U^k(y) &= \sum_j g^{kj} U_j(y) \ . \end{aligned} \tag{12}$$

In terms of column vectors we can write

$$\mathbf{U} = \begin{pmatrix} U_1(x) \\ U_2(x) \\ U_3(x) \end{pmatrix} = \begin{pmatrix} U^1(x) \\ U^2(x) \\ U^3(x) \end{pmatrix} \ .$$

Let $\mathbf{U}_\tau$ y $\mathbf{U}^\eta$ be the column vectors defined, respectively, by the covariant and contravariant componenets of $\mathbf{U}$,

$$\mathbf{U}_\tau = \begin{pmatrix} U_1(y) \\ U_2(y) \\ U_3(y) \end{pmatrix} \qquad \mathbf{U}^\eta = \begin{pmatrix} U^1(y) \\ U^2(y) \\ U^3(y) \end{pmatrix} \ . \tag{13}$$

The relationships (12) take the form

$$\mathbf{U}_\tau = \mathbb{G} \ \ \mathbf{U}^\eta \qquad \mathbf{U}^\eta = \mathbb{G}^{-1} \mathbf{U}_\tau , \tag{14}$$

and the transformation laws (11) and (10) are

$$\mathbf{U} = \mathbb{J} \ \mathbf{U}^\eta \ = \left(\mathbb{J}^{-1}\right)^t \ \mathbf{U}_\tau \ . \tag{15}$$

The relationship (7) between τ_j and η^k yields a simple expression for the inner product $\mathbf{U} \cdot \mathbf{W}$,

$$\mathbf{U} \cdot \mathbf{W} = \sum_k \eta^k W_k \cdot \sum_j \tau_j U^j = \sum_j W_j(y) U^j(y) \ ,$$

that in terms of the column vectors $\mathbf{W}_\tau$, $\mathbf{U}^\eta$ takes the form

$$\mathbf{U} \cdot \mathbf{W} = \mathbf{W}_\tau^t \ \mathbf{U}^\eta \ . \tag{16}$$

Let us now consider the calculation of the functional

$$F_S(\mathbf{W}) = \left\| \mathbf{W} - \mathbf{V}^0 \right\|_{DS} = \int_D \left\| \mathbf{W} - \mathbf{V}^0 \right\|_S^2 \ dx$$

where $\|\cdot\|_S$ is the norm associated to the inner product

$$\langle \mathbf{W} | \mathbf{U} \rangle_S = \mathbf{W}^t \mathbb{S} \ \mathbf{U}$$

and $\mathbf{W}$, $\mathbf{U}$ are the column vectors defined by the cartesian components. To get the expression of $F_S(\mathbf{W})$ in the coordinate system y we have the rule of change of variable for integrals

$$\int_D f(x) \ dx = \int_{Dy} f[x(y)] \sqrt{g} dy$$

where D_y is the image in the $y-$space of the original region D under the transformation $y^j = y^j(x)$. The inner product $\langle \mathbf{W} | \mathbf{U} \rangle_S$ is easily obtained in terms of the contravariant components $U^m(y)$, $W^i(y)$ of $\mathbf{U}$ and $\mathbf{W}$ using (15), namely,

$$\begin{aligned} \langle \mathbf{U} | \mathbf{W} \rangle_S &= \mathbf{U}^t \ \mathbb{S} \ \mathbf{W} \\ &= \left(\mathbb{J} \ \mathbf{U}^\eta\right)^t \ \ \mathbb{S} \ \mathbb{J} \ \mathbf{W}^\eta \\ &= \left(\mathbf{U}^\eta\right)^t \ \ \mathbb{J}^t \ \mathbb{S} \ \mathbb{J} \ \mathbf{W}^\eta \\ &= \left(\mathbf{U}^\eta\right)^t \ \ \mathbb{M} \ \mathbf{W}^\eta \end{aligned} \tag{17}$$

where we define

$$\mathbb{M} \equiv \mathbb{J}^t \ \mathbb{S} \ \mathbb{J} \ .$$

The matrix $\mathbb{M}$ can depend of each point $y = (y^1, y^2, y^3)$ in D_y, is symmetric and it is easy to show that if $\mathbb{S}(x)$ is positive defined, then $\mathbb{M}(y)$ also is. In this way the right side of (17) defines an inner product in the y space that we denote by $< \cdot | \cdot >_M$,

$$\langle \mathbf{U} | \mathbf{W} \rangle_M \equiv \left(\mathbf{U}^\eta\right)^t \ \ \mathbb{M} \ \mathbf{W}^\eta \ . \tag{18}$$

Thus we have the identity

$$\langle \mathbf{U} | \mathbf{W} \rangle_S = \langle \mathbf{U} | \mathbf{W} \rangle_M \ \ .$$

and the corresponding norms

$$\|\mathbf{U}\|_S = \langle \mathbf{U} | \mathbf{U} \rangle_S^{1/2} \qquad \|\mathbf{U}\|_M \equiv \langle \mathbf{U} | \mathbf{U} \rangle_M^{1/2}$$

satisfy

$$\|\mathbf{U}\|_S^2 = \|\mathbf{U}\|_M^2 \ \ . \tag{19}$$

The integration of (19) and the change of x by y yield

$$\|\mathbf{U}\|_{DS}^2 = \int_D \|\mathbf{U}\|_S^2 \ dx = \int_{Dy} \|\mathbf{U}\|_M^2 \ \sqrt{g} dy$$

and, therefore,

$$\|\mathbf{U}\|_{DS}^2 = \int_{Dy} \|\mathbf{U}\|_M^2 \ \sqrt{g} \ dy \ .$$

The right side defines a norm that we denote by $\parallel \mathbf{U} \parallel^2_{DM}$,

$$\parallel \mathbf{U} \parallel^2_{DM} \equiv \int_{Dy} \|\mathbf{U}\|^2_M \sqrt{g}\, dy \ . \tag{20}$$

The norm $\parallel \mathbf{U} \parallel_{DM}$ is the contravariant form of $\|\mathbf{U}\|^2_{DS}$ and the desired contravariant form of the functional $F_S(\mathbf{W})$ is

$$\begin{aligned} F_S(\mathbf{W}) &= \parallel \mathbf{W} - \mathbf{V}^0 \parallel^2_{DM} \\ &= \int_{Dy} \left\|\mathbf{W} - \mathbf{V}^0\right\|^2_M \sqrt{g}\, dy \\ &= \int_{Dy} \left\langle \mathbf{W} - \mathbf{V}^0 | \mathbf{W} - \mathbf{V}^0 \right\rangle_M \sqrt{g}\, dy \end{aligned} \tag{21}$$

2.3 Calculation of V with the contravariant form of $F_S(\mathbf{W})$

The space $\mathbb{V}$ that contains the desired adjusted field $\mathbf{V}$ was defined in x space,

$$\mathbb{V} = \{\mathbf{W}(x) : \nabla \cdot \mathbf{W} = 0 \text{ in } D,\ \mathbf{W} \cdot \mathbf{n} = 0 \text{ on } \partial D_b\}.$$

To compute $\mathbf{V}$ in the y space we use the contravariant form of $F_S(\mathbf{W})$ since the expression of $\nabla \cdot \mathbf{W} = 0$ is simple in terms of the contravariant components $W^k(y)$ of $\mathbf{W}$, namely,

$$\nabla \cdot \mathbf{W} = \frac{1}{\sqrt{g}} \sum_l \frac{\partial}{\partial y^l} \sqrt{g} W^l(y) \ .$$

The use of the contravariant form of each vector $\mathbf{W}$ in $\mathbb{V}$ suggests that the vector $\mathbf{n}$ normal to the lower boundary ∂D_b, should be represented in its covariant form,

$$\mathbf{n} = \sum_l n_l \eta^l$$

to get a simple expression of the inner product $\mathbf{W} \cdot \mathbf{n}$ [eq. (16)]

$$\mathbf{W} \cdot \mathbf{n} = \mathbf{n}^t_\tau \mathbf{W}^\eta$$

where we use the notation of column vectors, $\mathbf{n}_\tau = (n_1, n_2, n_3)^t$ y $\mathbf{W}^\eta = (W^1, W^2, W^3)^t$. In this way the lower boundary condition takes the form

$$\mathbf{n}^t_\tau \mathbf{W}^\eta = 0 \qquad \text{on } \partial D_{yb} \ . \tag{22}$$

Let D_y, ∂D_y, $\partial D_{ya}, \partial D_{yb}$ the imagine of D, ∂D, $\partial D_a, \partial D_b$ under the transformation $y^k = y^k(x)$, then ∂D_y has the decomposition

$$\partial D_y = \partial D_{ya} \cup \partial D_{yb} \ .$$

Accordingly, the space $\mathbb{V}$ can be defined as followsin

$$\mathbb{V} = \Big\{ \mathbf{W} = \sum_k \tau_k W^k(y) : \nabla \cdot \mathbf{W} = 0 \text{ in } D_y \quad \mathbf{n}^t_\tau \mathbf{W}^\eta = 0 \text{ on } \partial D_{by} \Big\}$$

Let us consider the calculation of the adjusted field $\mathbf{V}$. To simplify the notation we shall omit η and τ in $\mathbf{W}^\eta$ and $\mathbf{n}_\tau$. Suppose that there exists a field $\mathbf{V}$ that (in its contravariant form) minimizes $F_S(\mathbf{W})$ in the space $\mathbb{V}$, then it satisfies

$$\frac{d}{d\epsilon} F_S\left(\mathbf{V} + \epsilon \mathbf{W}\right)|_{\epsilon=0} = 0 \quad \text{for all} \quad \mathbf{W} \in \mathbb{V} \ .$$

Let

$$\Delta \mathbf{V} \equiv \mathbf{V} - \mathbf{V}^0.$$

According to (21) we have

$$\begin{aligned} F_S\left(\mathbf{V} + \epsilon \mathbf{W}\right) &= \int_{Dy} \left\langle \Delta \mathbf{V} + \epsilon \mathbf{W} | \Delta \mathbf{V} + \epsilon \mathbf{W} \right\rangle_M \sqrt{g}\, dy \\ &= \int_{Dy} \Big\{ \left\langle \Delta \mathbf{V} | \Delta \mathbf{V} \right\rangle_M + 2\epsilon \left\langle \Delta \mathbf{V} | \mathbf{W} \right\rangle_M \\ &\quad + \epsilon^2 \left\langle \mathbf{W} | \mathbf{W} \right\rangle_M \Big\} \sqrt{g}\, dy \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{d}{d\epsilon} F_S\left(\mathbf{V} + \epsilon \mathbf{W}\right)|_{\epsilon=0} &= 2 \int_{Dy} \left\langle \Delta \mathbf{V} | \mathbf{W} \right\rangle_M \sqrt{g}\, dy \\ &= 2 \int_{Dy} \Delta \mathbf{V}^t\, \mathbb{M} \mathbf{W} \sqrt{g}\, dy \\ &= 0 \end{aligned}$$

or, equivalently,

$$\int_{Dy} \Delta \mathbf{V}^t\, \mathbb{M}\, \mathbf{W} \sqrt{g}\, dy = 0 \ . \tag{23}$$

On the other hand, multiplying the condition $\nabla \cdot \mathbf{W} = 0$ by a function $\lambda(y)$ and integrating one gets

$$\int_{D_y} \lambda \nabla \cdot \mathbf{W} \sqrt{g}\, dy = 0 \ ,$$

and using the identity $\lambda \nabla \cdot \mathbf{W} = \nabla \cdot (\lambda \mathbf{W}) - \nabla \lambda \cdot \mathbf{W}$ we have

$$\int_{D_y} \left[\nabla \cdot (\lambda \mathbf{W}) - \nabla \lambda \cdot \mathbf{W}\right] \sqrt{g}\, dy = 0 \ . \tag{24}$$

The divergence theorem in the y space yields

$$\int_{D_y} \left(\nabla \cdot \lambda \mathbf{W}\right) \sqrt{g} dy = \oint_{\partial D_y} \lambda \hat{\mathbf{n}}^t_y \mathbf{W} \sqrt{g}\, ds_y$$

where $\hat{n}_{yl}$ are the components of the unit vector $\hat{\mathbf{n}}_y$ that is normal and out of the region ∂D_y, and $\hat{\mathbf{n}}_y$ denotes the corresponding column vector. Thus (24) takes the form

$$\oint_{\partial D_y} \lambda\ \hat{\mathbf{n}}_y^t \mathbf{W}\ \sqrt{g}\, ds_y - \int_{D_y} \nabla\lambda \cdot \mathbf{W}\sqrt{g}\, dy = 0\ .$$

The sum of this equation with (23) implies that $\Delta\mathbf{V}$ has to satisfy the integral equation

$$\int_{D_y} \left[\Delta\mathbf{V}^t \mathbb{M}\mathbf{W} - (\nabla\lambda)^t\, \mathbf{W}\right] \sqrt{g} dy + \oint_{\partial D_y} \lambda \hat{\mathbf{n}}_y^t \mathbf{W}\sqrt{g} ds_y\ = 0. \tag{25}$$

for all $\mathbf{W} \in \mathbb{V}$. In particular, if $\mathbf{W}$ is zero on the boundary ∂D_y one gets

$$\int_{D_y} \left[\Delta\mathbf{V}^t\, \mathbb{M} - (\nabla\lambda)^t\right]\ \mathbf{W}\sqrt{g}\, dy = 0\ .$$

The components $W^l(y)$ of $\mathbf{W}$ are not independent since they have to satisfy

$$\nabla \cdot \mathbf{W} = \frac{1}{\sqrt{g}} \sum_l \frac{\partial}{\partial y^l} \sqrt{g} W^l(y) = 0\ .$$

Suppose that W^k is the dependent component. If we impose to $\lambda(y)$ the condition that eliminates the term corresponding to W^k,

$$\sum_l \Delta V^l\ \mathbb{M}_{kl} - \frac{\partial\lambda}{\partial y^k} = 0 \qquad \text{for} \qquad k = k'$$

the integral equation (25) becomes

$$\int_D \sum_{k\neq k'} \left[\sum_l \left(\Delta V^l \mathbb{M}_{kl} - \frac{\partial\lambda}{\partial y^k}\right)\right] W^k dx = 0\ .$$

and since the components W^k $(k' \neq k)$ are independent we conclude that their coefficients satisfy

$$\sum_l \left(\Delta V^l\ \mathbb{M}_{kl} - \frac{\partial\lambda}{\partial y^k}\right) = 0 \qquad \text{para } k \neq k'\ .$$

In summary, if $\mathbf{V}$ minimizes $F_S(\mathbf{W})$ in the space $\mathbb{V}$, then the contravariant components of $\Delta\mathbf{V}$ satisfy the Euler-Lagrange equations

$$\mathbb{M}\Delta\mathbf{V} - \nabla\lambda = 0$$

where $\nabla\lambda$ denotes the column vector

$$\nabla\lambda = \begin{pmatrix} \partial\lambda/\partial y^1 \\ \partial\lambda/\partial y^2 \\ \partial\lambda/\partial y^3 \end{pmatrix}\ .$$

Therefore, $\Delta\mathbf{V} = \mathbb{M}^{-1}\nabla\lambda$ and $\mathbf{V}$, λ have to satisfy the relationship

$$\mathbf{V} = \mathbf{V}^0 + \mathbb{M}^{-1}\ \nabla\lambda, \tag{26}$$

which in terms of components takes the form

$$V^k = V^{0,k} + \sum_l \mathbb{M}_{kl}^{-1}\ \frac{\partial\lambda}{\partial y^l}\ . \tag{27}$$

Since the equation (26) does not depend of a particular field $\mathbf{W}$ in $\mathbb{V}$, it is valid for all $\mathbf{W}$'s in $\mathbb{V}$ and consequently the integral equation (25) becomes

$$\oint_{\partial Dy} \lambda\ \hat{\mathbf{n}}_y^t\ \mathbf{W}\ \sqrt{g}\, ds_y = 0 \text{ for } \mathbf{W} \in \mathbb{V}. \tag{28}$$

In this point we should remember that the boundary condition (22) is given in terms of the covariant components n_l of the vector $\mathbf{n}$ normal and exterior to the lower boundary ∂D_b. It is not hard to show that the component n_l is proportional to $\hat{n}_{yl}$; that is, there exists a function $c(y)$ such that

$$n_l = c(y)\hat{n}_{yl} \ \text{ or }\ \mathbf{n}_\tau = c(y)\hat{\mathbf{n}}_y \ \text{ on } \partial D_y \tag{29}$$

and the boundary condition (22) takes the form

$$\hat{\mathbf{n}}_y^t \mathbf{W} = 0 \quad \text{on} \quad \partial D_{yb}\ .$$

Hence the integral equation (28) becomes an integral equation that λ has to satisfy

$$\int_{\partial D_{ya}} \lambda\ \hat{\mathbf{n}}_y^t \mathbf{W}\sqrt{g}\, ds_y = 0 \quad \text{for all} \quad \mathbf{W} \in \mathbb{V}\ ,$$

Since $\hat{\mathbf{n}}_y^t\ \mathbf{W} \neq 0$ in general, we conclude that the last equation holds true only if λ satisfies the Dirichlet boundary condition

$$\lambda = 0 \quad \text{on} \quad \partial D_{ya}\ . \tag{30}$$

Thus, $\mathbf{V}$ minimizes the functional $F_S(\mathbf{W})$ only if $\mathbf{V}$ and λ satisfy (26) and λ satisfies the boundary condition (30). These conditions together with the additional conditions $\nabla \cdot \mathbf{V} = 0$ and $\hat{\mathbf{n}}_y^t \mathbf{V} = 0$ on ∂D_{yb} determine uniquely λ. In fact, the substitution of (27) into $\nabla \cdot \mathbf{V} = 0$ yields

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial y^k} \sqrt{g} V^k(y) \\ &= \nabla \cdot \mathbf{V}^0 + \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial y^k} \sqrt{g} \sum_l \mathbb{M}_{kl}^{-1}\ \frac{\partial\lambda}{\partial y^l} \\ &= 0\ , \end{aligned}$$

Hence we get the elliptic equation

$$L_y\lambda = \sqrt{g}\nabla \cdot \mathbf{V}^0 \quad \text{ in } D_y \tag{31}$$

where we define

$$L_y \equiv -\sum_{kl} \frac{\partial}{\partial y^k} \sqrt{g} \mathbb{M}^{-1}_{kl} \frac{\partial}{\partial y^l} = -\nabla^t \sqrt{g} \mathbb{M}^{-1} \nabla \ .$$

If we put (26) into the boundary condition $\hat{\mathbf{n}}^t_y \mathbf{V} = 0$,

$$\hat{\mathbf{n}}^t_y \mathbf{V} = \hat{\mathbf{n}}^t_y \left(\mathbf{V}^0 + \mathbb{M}^{-1} \nabla \lambda \right) = 0,$$

we get the Neumann boundary condition

$$\mathcal{L}_y \lambda = -\hat{\mathbf{n}}^t_y \mathbf{V}^0 \qquad \text{on} \quad \partial D_{yb} \tag{32}$$

where we define

$$\mathcal{L}_y \equiv \sum_{kl} \hat{n}_{yk} \ \mathbb{M}^{-1}_{kl} \frac{\partial}{\partial y^l} = \hat{\mathbf{n}}^t_y \ \mathbb{M}^{-1} \ \nabla \ .$$

As is shown in the next section, the mixed boundary conditions (30), (32) together with the elliptic equation (31) determine uniquely to λ and, therefore, the adjusted field $\mathbf{V}$.

Having computed λ we use (26) to obtain the contravariant components of the adjusted field $\mathbf{V}$. According to the transformation law (15), the product of (26) with the Jacobian matrix $\mathbb{J}$,

$$\mathbb{J}\mathbf{V}(y) = \mathbb{J} \left[\mathbf{V}^0(y) + \mathbb{M}^{-1} \ \nabla \lambda \right] ,$$

yields the cartesian components of the adjusted field,

$$\begin{aligned} \mathbf{V}(x) &= \mathbf{V}^0(x) + \mathbb{J} \ \mathbb{M}^{-1} \nabla \lambda \qquad (33) \\ &= \mathbf{V}^0(x) + \mathbb{S}^{-1} \left(\mathbb{J}^t \right)^{-1} \ \nabla \lambda \ , \end{aligned}$$

where we use $\mathbb{M}^{-1} = \left(\mathbb{J}^t \mathbb{S} \mathbb{J} \right)^{-1} = \mathbb{J}^{-1} \mathbb{S}^{-1} \left(\mathbb{J}^t \right)^{-1}$.

2.4 Some particular cases of $\mathbb{S}$ and coordinates y^i

The most common matrix S used in the literature is [1-8] is the diagonal matrix

$$\mathbb{S}_{kl} = \delta_{kl} \alpha_k^2$$

whose inverse is $\mathbb{S}^{-1}_{kl} = \delta_{kl} \alpha_k^{-2}$. In cartesian coordinates the elliptic problem for λ is

$$\begin{aligned} L\lambda &= \nabla \cdot \mathbf{V}^0 && (34) \\ \lambda &= 0 \qquad \text{for } x \in \partial D_a && (35) \\ \mathcal{L}\lambda &= -\mathbf{V}^0 \cdot \hat{\mathbf{n}} \qquad \text{for } x \in \partial D_b \ , && (36) \end{aligned}$$

where

$$L = -\nabla^t \mathbb{S}^{-1} \nabla = -\sum_k \frac{\partial}{\partial x^k} \frac{1}{\alpha_k^2} \frac{\partial}{\partial x^k}$$

$$\mathcal{L} = \hat{\mathbf{n}}^t \mathbb{S}^{-1} \nabla = \sum_k \frac{\hat{n}_k}{\alpha_k^2} \frac{\partial}{\partial x^k}$$

and the adjusted field $\mathbf{V}$ is given by

$$V^k = V^{0k} + \frac{1}{\alpha_k^2} \frac{\partial \lambda}{\partial x^k} \quad \text{for } k = 1, 2, 3$$

Almost all the terrain-following coordinates used in the literature keep the horizontal coordinates

$$y^1 = x \qquad y^2 = y$$

and only the vertical coordinate is changed. For instance, we have

$$\begin{aligned} y^3 &= \frac{H - z}{H - h(x,y)} \text{ with } h(x,y) \le z \le H(x,y) \\ z &= H + [h(y^1, y^2) - H] y^3 \qquad \text{[ref. (5)]}, \end{aligned}$$

$$\begin{aligned} y^3 &= \frac{z - h(x,y)}{H(x,y) - h(x,y)} \text{ with } h(x,y) \le z \le H(x,y) \\ z &= y^3 [H(y^1, y^2) - h(y^1, y^2)] + h(y^1, y^2) \qquad \text{[ref. (6)]}, \end{aligned}$$

$$\begin{aligned} y^3 &= \frac{z - h(x,y)}{\Delta H} \text{ with } \quad h(x,y) \le z \le \Delta H + h(x,y) \\ z &= y^3 \Delta H + h(y^1, y^2) \qquad \text{[ref. (4)]}, \end{aligned}$$

$$\begin{aligned} y^3 &= \frac{z_t - z}{z_t - z_s(x,y)} \quad \text{with} \quad z_s(x,y) \le z \le z_t \\ z &= z_t - y^3 \left[z_t - z_s(y^1, y^2) \right] \qquad \text{[ref. (7)]}, \end{aligned}$$

$$\begin{aligned} y^3 &= H \frac{z - h(x,y)}{H - h(x,y)} + z_0 \text{ with } \quad h(x,y) \le z \le H \\ z &= \frac{H - h(y^1, y^2)}{H} (y^3 - z_0) + h(y^1, y^2) \qquad \text{[ref. (10)]}. \end{aligned}$$

In each case the coordinates y^i have the form

$$y^1 = x \qquad y^2 = y \qquad y^3 = y^3(x, y, z)$$

or

$$x = y^1 \qquad y = y^2 \qquad z = z(y^1, y^3, y^2) \tag{37}$$

where the so-called sigma coordinate $\sigma = y^3$ is constant on the topography,

$$\sigma[x, y, z = h(x,y)] = cte \quad \text{on } \partial D_b \ ,$$

and the upper boundary $z = z_{max}$,

$$\sigma[x, y, z = z_{max}] = cte \quad \text{on } \partial D_a \ .$$

The matrices $\mathbb{J}$, $\mathbb{J}^{-1}$, $\mathbb{G}$, $\mathbb{G}^{-1}$ associated to the transformation (37) have the form

$$\begin{aligned}
\mathbb{J} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z_1 & z_2 & z_3 \end{pmatrix} \\
\mathbb{J}^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z_1/z_3 & -z_2/z_3 & 1/z_3 \end{pmatrix} \\
\mathbb{G} &= \begin{pmatrix} 1+z_1^2 & z_1 z_2 & z_1 z_3 \\ z_1 z_2 & 1+z_2^2 & z_2 z_3 \\ z_1 z_3 & z_2 z_3 & z_3^2 \end{pmatrix} \\
\mathbb{G}^{-1} &= \begin{pmatrix} 1 & 0 & -z_1/z_3 \\ 0 & 1 & -z_2/z_3 \\ -\frac{z_1}{z_3} & -\frac{z_2}{z_3} & \left(z_1^2 + z_2^2 + 1\right)/z_3^2 \end{pmatrix}
\end{aligned} \tag{38}$$

where we use the notation

$$z_j \equiv \frac{\partial z}{\partial y^j} \ , \qquad J = \sqrt{g} = z_3 = \frac{\partial z}{\partial y^3} \ .$$

We have

$$\mathbb{M} = \begin{bmatrix} \alpha_1^2 + z_1^2\alpha_3^2 & z_1 z_2 \alpha_3^2 & z_1 z_3 \alpha_3^2 \\ z_1 z_2 \alpha_3^2 & \alpha_2^2 + z_2^2 \alpha_3^2 & z_2 z_3 \alpha_3^2 \\ z_1 z_3 \alpha_3^2 & z_2 z_3 \alpha_3^2 & z_3^2 \alpha_3^2 \end{bmatrix} \ .$$

To compute $\mathbb{M}^{-1}$ we use $\mathbb{M}^{-1} = (\mathbb{J}^t \mathbb{S} \mathbb{J})^{-1} = \mathbb{J}^{-1}[(\mathbb{S}\mathbb{J})^{-1}]^t$ where

$$(\mathbb{S}\mathbb{J})^{-1} = \begin{bmatrix} 1/\alpha_1^2 & 0 & 0 \\ 0 & 1/\alpha_2^2 & 0 \\ -z_1/z_3\alpha_1^2 & -z_2/z_3\alpha_2^2 & 1/z_3\alpha_3^2 \end{bmatrix} ,$$

hence

$$\mathbb{M}^{-1} = \begin{bmatrix} \dfrac{1}{\alpha_1^2} & 0 & -\dfrac{z_1}{z_3\alpha_1^2} \\ 0 & \dfrac{1}{\alpha_2^2} & -\dfrac{z_2}{z_3\alpha_2^2} \\ -\dfrac{z_1}{z_3\alpha_1^2} & -\dfrac{z_2}{z_3\alpha_2^2} & M_{33}^{-1} \end{bmatrix} \tag{39}$$

where

$$M_{33}^{-1} = \left(\frac{z_1}{z_3\alpha_1}\right)^2 + \left(\frac{z_2}{z_3\alpha_2}\right)^2 + \left(\frac{1}{z_3\alpha_3}\right)^2 \ .$$

Using the standard notation for the cartesian components of the initial field $\mathbf{V}^0$

$$V^{1,0}(x) = u^0 \quad V^{2,0}(x) = v^0 \quad V^{3,0}(x) = w^0$$

or

$$\mathbf{V}^0 = \sum \hat{\mathbf{x}}_i V^{i,0}(x) = \hat{\mathbf{x}} u^0 + \hat{\mathbf{y}} v^0 + \hat{\mathbf{z}} w^0 \ .$$

the contravariant components of $\mathbf{V}^0$ are given by

$$\begin{pmatrix} V^{1,0}(y) \\ V^{2,0}(y) \\ V^{3,0}(y) \end{pmatrix} = \begin{pmatrix} u^0 \\ v^0 \\ \dfrac{-z_1 u^0 - z_2 v^0 + w^0}{z_3} \end{pmatrix} \ . \tag{40}$$

In a similar way, for the adjusted field

$$\mathbf{V} = \hat{\mathbf{x}} u + \hat{\mathbf{y}} v + \hat{\mathbf{z}} w$$

we have

$$\begin{pmatrix} V^1(y) \\ V^2(y) \\ V^3(y) \end{pmatrix} = \begin{pmatrix} u \\ v \\ \dfrac{-z_1 u - z_2 v + w}{z_3} \end{pmatrix} \ . \tag{41}$$

The continuity equation $\nabla \cdot \mathbf{V} = 0$ has the form

$$\frac{\partial}{\partial y^1} z_3 V^1 + \frac{\partial}{\partial y^2} z_3 V^2 + \frac{\partial}{\partial y^3} z_3 V^3 = 0$$

or

$$\frac{\partial}{\partial y^1} z_3 u + \frac{\partial}{\partial y^2} z_3 v + \frac{\partial}{\partial y^3}(-z_1 u - z_2 v + w) = 0 \ .$$

For the elliptic equation $L_y \lambda = \sqrt{g} \nabla \cdot \mathbf{V}^0$ (31) we have

$$\begin{aligned}
L_y &= -\nabla^t \sqrt{g} \mathbb{M}^{-1} \nabla = -\nabla^t \ z_3 \mathbb{M}^{-1} \ \nabla \qquad (42) \\
&= -(\partial_1 \ \partial_2 \ \partial_3) \begin{bmatrix} \dfrac{z_3}{\alpha_1^2}\partial_1 - \dfrac{z_1}{\alpha_1^2}\partial_3 \\ \dfrac{z_3}{\alpha_2^2}\partial_2 - \dfrac{z_2}{\alpha_2^2}\partial_3 \\ -\dfrac{z_1}{\alpha_1^2}\partial_1 - \dfrac{z_2}{\alpha_2^2}\partial_2 + z_3 M_{33}^{-1} \ \partial_3 \end{bmatrix}
\end{aligned}$$

or

$$\begin{aligned}
-L_y &= \partial_1 \ \frac{1}{\alpha_1^2}\left(z_3 \partial_1 - z_1 \ \partial_3\right) + \partial_2 \ \left(z_3 \ \partial_2 - z_2 \ \partial_3\right) \qquad (43) \\
&+ \partial_3 \ \left\{ -\frac{z_1}{\alpha_1^2} \ \partial_1 - \frac{z_2}{\alpha_2^2} \ \partial_2 + z_3 M_{33}^{-1} \ \partial_3 \right\} \ ,
\end{aligned}$$

where

$$z_3 M_{33}^{-1} = \frac{1}{z_3}\left[\left(\frac{z_1}{\alpha_1}\right)^2 + \left(\frac{z_2}{\alpha_2}\right)^2 + \left(\frac{1}{\alpha_3}\right)^2\right] \tag{44}$$

and

$$\begin{aligned}
\sqrt{g}\nabla \cdot \mathbf{V}^0 \quad &= \quad \frac{\partial}{\partial y^1} z_3 u^0 + \frac{\partial}{\partial y^2} z_3 v^0 \\
&+ \frac{\partial}{\partial y^3}\left(-z_1 u^0 - z_2 v^0 + w^0\right) \ . \qquad (45)
\end{aligned}$$

The Dirichlet boundary condition $\lambda = 0$ on ∂D_{ya} is equivalent to the boundary conditions

$$\begin{aligned} \lambda\left(y^1 = x_{\min}, y^2, y^3\right) &= 0 \ , \\ \lambda\left(y^1 = x_{\max}, y^2, y^3\right) &= 0 \ , \\ \lambda\left(y^1, y^2 = y_{\min}, y^3\right) &= 0 \ , \\ \lambda\left(y^1, y^2 = y_{\max}, y^3\right) &= 0 \ , \\ \lambda\left(y^1, y^2, y^3 = z_{\max}\right) &= 0 \ . \end{aligned} \tag{46}$$

For the Neumann boundary condition $\mathcal{L}_y\lambda = -\hat{\mathbf{n}}_y^t\mathbf{V}^0$ on $\partial D_{yb} = \left\{(y^1, y^2, y^3 = \sigma_{min})\right\}$ we have

$$\hat{\mathbf{n}}_y = \sum_l \hat{\mathbf{y}}_l \hat{n}_{yl} = -\hat{\mathbf{y}}_3 \ ,$$

and therefore

$$\begin{aligned} \mathcal{L}_y &= \hat{\mathbf{n}}_y^t \mathbb{M}^{-1}\nabla = \begin{pmatrix} 0, & 0, & -1 \end{pmatrix} \mathbb{M}^{-1} \nabla \\ &= \frac{z_1}{z_3\alpha_1^2}\ \partial_1 + \frac{z_2}{z_3\alpha_2^2}\ \partial_2 - M_{33}^{-1}\ \partial_3 \end{aligned} \tag{47}$$

and

$$\hat{\mathbf{n}}_y^t \mathbf{V}^0 = -V^{3,0} \ .$$

Thus $\mathcal{L}_y\lambda = -\hat{\mathbf{n}}_y^t\mathbf{V}^0$ takes the form

$$\left(\frac{z_1}{z_3\alpha_1^2}\ \partial_1 + \frac{z_2}{z_3\alpha_2^2}\ \partial_2 - M_{33}^{-1}\ \partial_3\right)\lambda = V^{3,0}(y) \ . \tag{48}$$

The contravariant components of the adjusted field are [eq. (26)]

$$\begin{aligned} V^1(y) &= V^{1,0}(y) + \frac{\partial_1\lambda}{\alpha_1^2} - \frac{z_1\partial_3\lambda}{z_3\alpha_1^2} \\ V^2(y) &= V^{2,0}(y) + \frac{\partial_2\lambda}{\alpha_2^2} - \frac{z_2\partial_3\lambda}{z_3\alpha_2^2} \\ V^3(y) &= V^{3,0}(y) - \frac{z_1\partial_1\lambda}{z_3\alpha_1^2} - \frac{z_2\partial_2\lambda}{z_3\alpha_2^2} + M_{33}^{-1}\partial_3\lambda \end{aligned} \tag{49}$$

and the Cartesian components are [eq. (33)]

$$\begin{aligned} V^1(x) &= V^{1,0}(x) + \frac{\partial_1\lambda}{\alpha_1^2} - \frac{z_1\partial_3\lambda}{z_3\alpha_1^2} \\ V^2(x) &= V^{2,0}(x) + \frac{\partial_2\lambda}{\alpha_2^2} - \frac{z_2\partial_3\lambda}{z_3\alpha_2^2} \\ V^3(x) &= V^{3,0}(x) + \frac{\partial_3\lambda}{z_3\alpha_3^2} \ . \end{aligned} \tag{50}$$

2.5 The boundary condition $\partial\lambda/\partial n = 0$ is incorrect if $\alpha_i^2 \neq 1$

The standard procedure used in the literature [1-8] to compute the adjusted field $\mathbf{V}$ with the functional

$$F(\mathbf{W}) = \int_D \sum_i \alpha_i^2 \left(W^i - V^{i,0}\right)^2\ dx$$

and the condition $\nabla\cdot\mathbf{V} = 0$ consists in minimizing the functional

$$J(\mathbf{W},\lambda) = \int_D \left[\sum_i \alpha_i^2 \left(W^i - V^{i,0}\right)^2 + \lambda\nabla\cdot\mathbf{W}\right]\ dx$$

It should be noted that no of the references [1-8] considered the definition of the space $\mathbb{V}$ that contains $\mathbf{V}$ and the test fields $\mathbf{W}$. This ambiguity is reflected by the boundary condition

$$\lambda\delta\mathbf{V}\cdot\hat{\mathbf{n}} = 0 \quad \text{on} \quad \partial D \tag{51}$$

obtained by minimizing $J(\mathbf{V}+\epsilon\delta\mathbf{V},\lambda)$. In fact, the equation (51) holds of either λ or $\delta\mathbf{V}\cdot\hat{\mathbf{n}}$ is zero on ∂D, since these conditions cannot be imposed simultaneously (or the problem is undetermined) the following choice was adopted by Sherman [3] and other authors:

1. The Dirichlet boundary condition
$$\lambda = 0 \tag{52}$$
is used for an open boundary ∂D.

2. The Neumann boundary condition
$$\frac{\partial\lambda}{\partial n} = \nabla\lambda\cdot\hat{\mathbf{n}} = 0 \tag{53}$$
is used to "impose a no-flow boundary condition on ∂D" [1-8]. It is generally accepted that such a boundary condition is suitable on the terrain

Thus, the elliptic problem solved by several authors has the form

$$\begin{aligned} \left(-\sum \frac{\partial}{\partial x^k}\frac{1}{\alpha_k^2}\frac{\partial}{\partial x^k}\right)\lambda &= \nabla\cdot\mathbf{V}^0 \quad \text{in} \quad D \\ \lambda &= 0 \quad \text{on} \quad \partial D_a \\ \frac{\partial\lambda}{\partial n} &= \nabla\lambda\cdot\hat{\mathbf{n}} \\ &= \sum_k \hat{n}_k\frac{\partial\lambda}{\partial x^k} = 0 \text{ on } \partial D_b \ , \end{aligned}$$

where, by simplicity, we consider expressions in cartesian coordinates. In previous sections we have shown that the no-flow boundary condition $\mathbf{V}\cdot\mathbf{n} = 0$ on the topography ∂D_b holds true only if λ satisfies the boundary condition

$$\sum_k \frac{\hat{n}_k}{\alpha_k^2}\frac{\partial\lambda}{\partial x^k} = 0 \quad \text{on} \quad \partial D_b$$

so that the condition (53) is incorrect when $\alpha_i^2 \neq 1$.

3 Analytic solutions of $\nabla \cdot \mathbf{V}(\mathbf{r}) = 0$

An important aspect of the present work to analyze the reliability of MCM's is the use of analytic solutions of the dynamic restrictions used by MCM's in their two-dimensional version on the $xz-$plane, namely,

$$\nabla \cdot \mathbf{V}_{\rm true} = 0 \tag{54}$$

where $\mathbf{V}_{\rm true} = u_{\rm true}\mathbf{i} + w_{\rm true}\mathbf{k}$ and the no-flow boundary condition

$$\mathbf{V}_{\rm true} \cdot \mathbf{n} = 0 \quad \text{on} \ \ z = h(x). \tag{55}$$

In this section we describe briefly the method used to obtain such exact solutions [12] .

To begin consider an abstract complex plane with variable

$$\zeta = \bar{x} + i\ \bar{z}\ .$$

In this plane we consider a uniform flow

$$\bar{V} = V_0 \qquad (\bar{u} = V_0,\, \bar{w} = 0)$$

obtained from the potential

$$\bar{\phi} = V_0 \bar{x}.$$

The physical space can be seen as the complex plane associated to the variable

$$\xi = x + i\ z\ .$$

Suppose that $h(\zeta)$ is an analytic function of ζ and let h_1 and h_2 be the real and imaginary parts of $h(\zeta)$,

$$h(\zeta) = h_1(\bar{x},\ \bar{z}) + i h_2(\bar{x},\ \bar{z})\ .$$

Then the function

$$G(\zeta) = \zeta + i h(\zeta)$$

is also an analytic function of ζ and defines the transformation equations

$$\begin{aligned} x &= x(\bar{x},\ \bar{z}) = \bar{x} - h_2(\bar{x},\ \bar{z}) \\ z &= z(\bar{x},\ \bar{z}) = \bar{z} + h_1(\bar{x},\ \bar{z}). \end{aligned}$$

It is clear that the imagine of the real axis $\bar{z} = 0$ in the $\zeta-$plane under these transformation equations is the curve that represents to the topography,

$$\{(x, h(x)\} = G[\{\bar{x}, \bar{z} = 0\}];$$

that is, we have

$$x = \bar{x}, \qquad z = h(\bar{x})\ .$$

Since the real axis $\bar{z} = 0$ is a stream line of the flow $\bar{V}$, the curve $z = h(\bar{x})$ is a stream line of the flow V that is the image of $\bar{V}$ under the transformation G. The components of the flow $\mathbf{V}_{\rm true} = u_{\rm true}\mathbf{i} + w_{\rm true}\mathbf{k}$ are

$$\begin{aligned} u_{\rm true} &= \frac{V_0}{J}\frac{\partial\ z(\bar{x},\ \bar{z})}{\partial\ \bar{z}} \\ w_{\rm true} &= -\frac{V_0}{J}\frac{\partial\ x(\bar{x},\ \bar{z})}{\partial\ \bar{z}} \end{aligned}$$

where

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial \bar{z}} \\ \frac{\partial z}{\partial \bar{x}} & \frac{\partial z}{\partial \bar{z}} \end{pmatrix}$$

and, since $G(\zeta)$ is analytic, the Cauchy-Riemann equations hold,

$$\frac{\partial x}{\partial \bar{x}} = \frac{\partial z}{\partial \bar{z}} \qquad \frac{\partial x}{\partial \bar{z}} = -\frac{\partial z}{\partial \bar{x}}\ .$$

Inherent problems of the map conforming do not permit us the direct use of $h(x)$. These problems are solved by a simpler representation of $h(x)$, namely, a *natural spline* $S(x)$ which is defined as follows. Let $\{x_k\}_{k=0}^{n}$ be a set of points where the terrain height $h(x_k)$ is known, then : (i) $S(x)$ satisfies

$$S(x_k) = h(x_k) \text{ for } k = 0, ..., n,$$

(ii) $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$,

$$\begin{aligned} S(x) = a_k &+ b_k(x - x_k) + c_k(x - x_k)^2 \\ &+ d_k(x - x_k)^3 \text{ for } x \in [x_k, x_{k+1}]\ , \end{aligned}$$

(iii) $S(x)$ and its derivatives $S'(x)$, $S''(x)$ are continuous on $[x_0, x_n]$ and $S''(x)$ satisfies

$$S''(x_0) = S''(x_n) = 0.$$

There is a unique natural spline associated to an analytic function $h(x)$ on the interval $[x_0, x_n]$. Since $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$, we can compute the flow

$$u_{\rm true} = \frac{V_0}{J}\frac{\partial z^{(k)}}{\partial \bar{z}}\ ,\ \ w_{\rm true} = -\frac{V_0}{J}\frac{\partial x^{(k)}}{\partial \bar{z}} \text{ for } x \in [x_k, x_{k+1}],$$

where

$$\begin{aligned} x^{(k)} &= \bar{x} - S_2(\bar{x},\ \bar{z}) \\ z^{(k)} &= \bar{z} + S_1(\bar{x},\ \bar{z}) \end{aligned}$$

and

$$S(\zeta = \bar{x} + i\bar{z}) = S_1(\bar{x},\ \bar{z}) + i\ S_2(\bar{x},\ \bar{z}).$$

The continuity of $S(x)$, $S'(x)$ and $S''(x)$ guarantees that the field $\mathbf{V}_{\rm true} = u_{\rm true}\mathbf{i} + w_{\rm true}\mathbf{k}$, its first derivatives

$$\frac{\partial u_{\rm true}}{\partial x},\ \frac{\partial u_{\rm true}}{\partial z},\ \frac{\partial w_{\rm true}}{\partial x},\ \frac{\partial w_{\rm true}}{\partial z},$$

and $\nabla \cdot \mathbf{V}_{\rm true}$ are continuous on the interval $[x_0, x_n]$. This together with the fact that $u_{\rm true}$, $w_{\rm true}$ satisfy the continuity equation (54) and the boundary condition (55) on each interval $[x_k, x_{k+1}]$, implies that the field $\mathbf{V}_{\rm true}$ satisfies the same equations on the whole interval $[x_0, x_n]$. In the present work we use smoothing splines [12] to obtain a velocity field with a simpler structure, since natural splines yields fields with a very complex structure as is shown in [12].

4 Numerical Calculation of λ

4.1 Variational formulation of the problem $L_y\lambda = f$

In this section we consider the variational formulation (see, e.g., [13]) of the elliptic equation (31),

$$L_y\lambda = f \quad \text{in} \quad D_y \tag{56}$$

with the mixed boundary conditions

$$\lambda = 0 \quad \text{on} \quad \partial D_{ya} \ , \tag{57}$$

$$\mathcal{L}_y\lambda = q \quad \text{on} \quad \partial D_{yb} \, , \tag{58}$$

where $\partial D_y = \partial D_{ya} \cup \partial D_{yb}$ is the boundary of the region D_y in the y space, and L_y, $\mathcal{L}_y$, f, q are given by

$$\begin{array}{lll} L_y &=& -\nabla^t\sqrt{g}\mathbb{M}^{-1}\nabla \qquad f = \sqrt{g}\nabla\cdot\mathbf{V}^0 \\ \mathcal{L}_y &=& \hat{\mathbf{n}}_y^t\mathbb{M}^{-1}\nabla \qquad\quad q = -\hat{\mathbf{n}}_y^t\mathbf{V}^0 \ . \end{array}$$

The solution λ is (at least) square integrable so that it belongs to the space

$$L_2(D_y) = \left\{ v : \int_{D_y} |v|^2 \ dy < \infty \right\}$$

endowed with the inner product

$$\langle v|u\rangle_y \equiv \int_{D_y} v\, u\, dy \ ,$$

and the norm $\parallel v \parallel_y = \langle v|v\rangle_y^{1/2}$. The function λ also belongs to the space of functions that satisfy the Dirichlet boundary condition (57), a space that we denote by V_a. In what follows we consider that V_a is complete with the norm

$$\|v\|_{y1}^2 \equiv \|v\|_y^2 + \sum_k \left\| \frac{\partial\, v}{\partial y^k} \right\|_y^2 \ .$$

In order to obtain the variational (or weak) formulation of the boundary value problem (56-58) the equation (56) is multiplied by a function v in V_a and integrated, thus

$$\langle L_y\, \lambda|v\rangle_y = \langle f|v\rangle_y \quad \text{holds for all } v\, \in V_a \ . \tag{59}$$

The left side is rewritten by integrating by parts

$$\int_{Dy} \frac{\partial\, w}{\partial y^k} v\, dy = \oint_{\partial Dy} v w \hat{n}_{yk}\, ds_y - \int_{Dy} w \frac{\partial\, v}{\partial y^k}\, dy \ ,$$

where $\hat{n}_{yk}$ are the components of the vector normal, unitary and exterior to the boundary ∂D_y. Making calculations we get

$$\begin{aligned} \langle L_y\lambda|v\rangle_y &= -\sum_{kl} \left\langle \frac{\partial}{\partial y^k}\sqrt{g}M_{kl}^{-1}\frac{\partial\,\lambda}{\partial y^l}|v \right\rangle_y \\ &= -\oint_{\partial D_y} \sqrt{g} v \mathcal{L}_y \lambda\, ds_y + a_y(\lambda, v)\, , \end{aligned}$$

where

$$\mathcal{L}_y = \sum_{kl} \hat{n}_{yk} M_{kl}^{-1} \frac{\partial}{\partial y^l} \ ,$$

In terms of the bilinear form

$$\begin{aligned} a_y(u, v) \ &\equiv \ \sum_{kl} \left\langle \sqrt{g} M_{kl}^{-1} \frac{\partial\, u}{\partial y^l}, \frac{\partial\, v}{\partial y^k} \right\rangle_y \\ &= \ \int_{D_y} (\nabla u)^t \ \ \mathbb{M}^{-1} \nabla v \sqrt{g}\ dy \end{aligned}$$

the eq. (59) has the form

$$a_y(\lambda, v) = \langle f|v\rangle_y + \oint_{\partial D_y} \sqrt{g} v \mathcal{L}_y \lambda\, ds_y \text{ for all } v \in V_a.$$

Since v satisfies $v = 0$ on ∂D_{ya} and $\mathcal{L}_y\lambda = q$ the integral on ∂D_y takes the form

$$\oint_{\partial D_y} \sqrt{g} v \mathcal{L}_y \lambda\, ds_y = \int_{\partial D_{yb}} \sqrt{g} v q\, ds_y \ .$$

Thus, we conclude that if the function λ exists it has to satisfy the integral equation

$$a_y(\lambda, v) = \tilde{f}_y(v) \quad \text{for all } v \in V_a \tag{60}$$

where we define

$$\tilde{f}_y(v) \equiv \langle f|v\rangle_y + \int_{\partial D_{yb}} \sqrt{g} v q\, ds_y \ .$$

The equation (60) is called the weak form of the elliptic problem (56-58).

The existence and uniqueness of λ is guaranteed by the properties of $a_y(\cdot,\cdot)$ y $\tilde{f}_y(\cdot)$. It is easy to see that $a_y(\cdot,\cdot)$ defines an inner product in the space V_a since $a_y(\cdot,\cdot)$ is

1. symmetric, $a_y(u, v) = a_y(v, u)$,

2. positive defined , $a_y(u,u) \geq 0$, and $a_y(u,u) = 0$ only for $u = 0$,

3. bilineal, $a_y(u, c_1 v + c_2 w) = c_1 a_y(u,v) + c_2 a_y(u,w)$.

In fact, using the symmetry of $\mathbb{M}^{-1}$ we get

$$(\nabla u)^t \ \mathbb{M}^{-1} \ \nabla v = (\nabla v)^t \ \mathbb{M}^{-1} \ \nabla u \ ,$$

and therefore

$$\begin{aligned} a_y(u,v) &= \int_{D_y} (\nabla u)^t \ \mathbb{M}^{-1} \ \nabla v \sqrt{g} \ dy \\ &= \int_{D_y} (\nabla v)^t \ \mathbb{M}^{-1} \ \nabla u \sqrt{g} \ dy = a_y(v,u) \ . \end{aligned}$$

If the transformation of coordinates $x^i = x^i(y)$ is well behaved and the coefficients $\alpha_i^2(y)$ are non zero on $\bar{D} \equiv D \cup \partial D$, then $\mathbb{M}^{-1}$ (y) is positive definite at each point in $\bar{D}$, that is,

$$\mathbf{U}^t \ \mathbb{M}^{-1} \ \mathbf{U} > 0 \text{ for } \mathbf{U} \neq \mathbf{0} \ .$$

In particular for $\mathbf{U} = \nabla u$ we have $(\nabla u)^t \ \mathbb{M}^{-1} \ \nabla u > 0$ when u is not the constant function. Multiplying the last inequality by$\sqrt{g}$ and integrating by parts we conclude that $a_y(\cdot,\cdot)$ is positive definite on V_a,

$$a_y(u,u) > 0 \text{ for all } u \in V_a \ .$$

The linearity of $a_y(\cdot,\cdot)$ is obvious. Finally, if $f(y)$ and $q(y)$ are piecewise continuos, the functional $\tilde{f}_y(\cdot)$ is bounded. According to the **Lax-Milgram** theorem [13] these properties guarantee the existence and uniqueness of the function λ that satisfies the integral equation (60).

The symmetry of $a_y(\cdot,\cdot)$ allows us to see λ as an extremal of the quadratic functional

$$J_y(v) = \frac{1}{2} a_y(v,v) - \tilde{f}_y(v) \quad \text{for } v \in V_a \ .$$

Let us remember that a function u is an extremal of the functional $J_y(v)$ if it satisfies

$$\frac{d}{d\epsilon} J_y(u + \epsilon v)|_{\epsilon=0} = 0 \text{ for all } v \in V_a \ .$$

According to this definition, let us show that the last equation is exactly the integral equation (60) to prove that λ is an extremal of $J_y(*)$. Using the symmetry and linearity of $a_y(\cdot,\cdot)$ and $\tilde{f}_y(\cdot)$ we get

$$\begin{aligned} J_y(\lambda + \epsilon v) &= \frac{1}{2} a_y(\lambda + \epsilon v, \lambda + \epsilon v) - \tilde{f}_y(\lambda + \epsilon v) \\ &= \frac{1}{2} a_y(\lambda,\lambda) + \epsilon a_y(\lambda, v) \\ &+ \frac{1}{2} \epsilon^2 a_y(v,v) - \tilde{f}_y(\lambda) - \epsilon \tilde{f}_y(v) \end{aligned}$$

The derivative at $\epsilon = 0$ yields

$$\frac{d}{d\epsilon} J_y(\lambda + \epsilon v)|_{\epsilon=0} = a_y(\lambda, v) - \tilde{f}_y(v) = 0 \ .$$

This is exactly the equation (60) and, therefore, a solution λ of (60) is an extremal of $J_y(v)$. The Lax-Milgram theorem guarantees that λ exits and is unique.

The above results suggest that the extremal λ can be compute by minimizing the functional $J(\lambda)$ instead of solving the integral equation (60). This way has the advantage of considering in an implicitly the Nuemann boundary condition (58) so that we have to consider only the Dirichlet boundary conditions (56). To prove this assertion let us show that if λ is an extremal of $J(\lambda)$, it automatically satisfies the Neumann boundary condition (58).

We have shown that if λ minimizes $J_y(\lambda)$, then satisfies (60). Let us consider (60) with v in $C_0^\infty(D_y)$. Since v is zero on the boundary ∂D_y, it belongs to the space V_a so that λ satisfies (60) for all $v \in C_0^\infty(D_y)$,

$$a_y(\lambda, v) = \langle f|v \rangle_y + \oint_{\partial D_{yb}} \sqrt{g} \ v \ q \, ds_y$$

but $v = 0$ on ∂D_y and hence the last equation becomes

$$a_y(\lambda, v) = \langle f|v \rangle_y \quad \text{for } v \in C_0^\infty(D_y) \ .$$

On the other hand, if λ has continuous second derivatives we can integrate by parts the left side to get

$$a_y(\lambda, v) = \langle L_y \lambda | v \rangle_y \quad \text{for } v \in C_0^\infty(D_y) \ .$$

Thus we get

$$\langle L_y \lambda - f | v \rangle_y = 0 \quad \text{for } v \in C_0^\infty(D_y) \ ,$$

and, therefore, λ satisfies the elliptic equation

$$L_y \lambda = f \quad \text{in } D_y \ . \tag{61}$$

If we now consider (60) in the form

$$\langle L_y \lambda | v \rangle_y + \int_{\partial D_{yb}} \sqrt{g} v \mathcal{L}_y \lambda \, ds_y = \langle f|v \rangle_y + \int_{\partial D_{yb}} \sqrt{g} v q \, ds_y$$

and we use that fact that λ satisfies (4.6), the terms $\langle L\lambda|v \rangle_y$ and $\langle f|v \rangle_y$ disappear so that λ satisfies

$$\int_{\partial D_{yb}} \sqrt{g} v \ \mathcal{L}_y \lambda \, ds_y = \int_{\partial D_{yb}} \sqrt{g} v \ q \, ds_y \text{ for all } v \in V_a$$

and since $v \neq 0$, in general, it follows that λ has to satisfy

$$\mathcal{L}_y \lambda = q \quad \text{on } \partial D_{yb} \ .$$

In other words, if λ minimizes $J_y(v)$, it satisfies automatically the Neumann Boundary condition (58). By this reason, such a condition is called *natural boundary condition.*

4.2 The finite element method

We choose the finite element method to solve the elliptic problem (34-36) because we want to work in cartesian coordinates. We have the following spaces of test functions

$$\begin{aligned} W &= H^1(D) \\ W_0 &= \{\phi \in H^1(D) : \phi = 0 \text{ on } \partial D_a\}. \end{aligned}$$

The integral equation (60) with $f = \nabla \cdot \mathbf{V}^0$ and $q = -\hat{\mathbf{n}}_y^t \mathbf{V}^0$ has the simple form

$$\int_D (\mathbb{S}^{-1} \nabla \lambda) \cdot \nabla \phi \ dx = - \int_D \mathbf{V}^0 \cdot \nabla \phi \text{ for all } \phi \in W_0 \ . \tag{62}$$

Let h be a discretization step and denote by J_h a finite element triangulation of $\bar{D}$. If P_1 is the space of polynomials in two variables of degree ≤ 1, then we approximate the function spaces W and W_0 by the finite dimensional spaces

$$\begin{aligned} W_h &= \left\{\phi_h \in C^0(\overline{D}) : \phi_h|_T \in P_1, \text{ for } T \in J_h\right\} \ , \\ W_{0h} &= \{\phi_h \in W_h : \phi_h = 0 \text{ on } \partial D_a\}, \end{aligned}$$

respectively. The finite element formulation of the problem (62) is: Find $\lambda_h \in W_{0h}$ such that

$$\int_D (\mathbb{S}^{-1} \nabla \lambda_h) \cdot \nabla \phi_h dx = - \int_D \mathbf{V}_h^0 \cdot \nabla \phi_h \text{ for all } \phi_h \in W_{0h} \ . \tag{63}$$

In this last equation $\mathbf{V}_h^0$ is the interpolant function on $W_h \times W_h$ that approximates the initial vector field $\mathbf{V}^0$.

Remark. It is well known [13] that the piecewise linear approximation λ_h is second order accurate.

Next, we will include a more detailed description of the finite element approximation (63). Let N be the total number of vertices in the triangulation J_h of $\overline{D}$, and suppose a numeration of those vertices has already been introduced. Thus, a basis of the finite dimensional space W_h is the collection of "hat functions" associated to those vertices:

$$\beta_h = \{\varphi_p\}_{1 \leq p \leq N}$$

where φ_p is a piece wise linear function such that

$$\varphi_p(Q) = \delta_{pQ} = \begin{cases} 1 \text{ if } p = Q \\ 0 \text{ if } p \neq Q \end{cases} \quad \text{for } Q \in \{1, ...N\} \ .$$

The support of these functions is the union of triangles in T_h that contain P as a vertex. In the same way, the basis of the finite dimensional space W_{0h} is

$$\begin{aligned} \beta_{0h} &= \{\varphi_p \in \beta_h : \varphi_p(Q) = 0 \text{ if } Q \text{ is a vertex on } \partial D_a\} \\ &= \{\varphi_p \in \beta_h : p = 1, 2, \ldots, N_0\} \end{aligned}$$

where N_0 is the number of vertices that do not belong to ∂D_a. Using these basis functions the solution λ_h can be expressed as

$$\lambda_h(x) = \sum_{i=1}^{N_0} \lambda(x_i) \varphi_i(x)$$

where $\{\mathbf{x}_i\}_{i=1}^{N_0}$ is the collection of vertices in the triangulation that do not belong to ∂D_a. If we denote $\lambda(\mathbf{x}_i)$ by λ_i the problem (63) is equivalent to the following linear algebraic problem: Find $\{\lambda_i\}_{i=1}^{N_0}$ in $\mathbb{R}^{N_0}$ such that

$$\sum_{i=1}^{N_0} a_{ij} \lambda_i = f_i \ , \ i = 1, \ldots, N_0 \ , \tag{64}$$

where

$$a_{ij} = \int_D \left(\mathbb{S}^{-1} \nabla \psi_j\right) \cdot \nabla \psi_i dx$$

and

$$f_i = - \int_D \mathbf{V}_h^0 \cdot \nabla \psi_i dx \ .$$

The matrix a_{ij} keeps the properties of $\mathbb{S}^{-1}$, namely, a_{ij} is symmetric and positive definite. The algebraic problem (64) can be solved by the conjugate gradient algorithm, in this work we use a conjugate gradient method adapted for sparse linear systems [15].

Once λ_h is computed, we finally compute an approximation $\mathbf{V}_h$ of the adjusted wind field $\mathbf{V}$ by

$$\mathbf{V}_h = \mathbf{V}_h^0 + \mathbb{S}^{-1} \nabla \lambda_h \ .$$

However, since λ_h is a piecewise linear function, $\nabla \lambda_h$ is constant on each triangle and is not defined on the edges of the triangulation. Thus we should compute $\mathbf{V}_h$ in the *weak* sense. Let $\mathbf{U}_h = \mathbf{V}_h - \mathbf{V}_h^0$, then $\mathbf{U}_h$ can be computed by solving the following problem: Find $\mathbf{U}_h$ such that

$$\int_D \mathbf{U}_h \cdot \mathbf{w} \ dx = \int_D \left(\mathbb{S}^{-1} \nabla \lambda_h\right) \cdot \mathbf{w} \ dx \text{ for all } \mathbf{w}$$

and $\mathbf{V}_h$ is given by $\mathbf{V}_h = \mathbf{U}_h + \mathbf{V}_h^0$. An easier alternative is to compute $\mathbf{U}_h = (u_1, u_3)$ pointwise: For $k = 1, 3$ find $u_k \in W_h$ such that

$$\int_D u_k \phi_i \ dx = \int_D \frac{1}{\alpha_i^2} \frac{\partial \lambda}{\partial x^k} \phi_i \ dx \text{ for all } i = 1, 2, \ldots, N \ .$$

The integral in the right hand side can be computed exactly by the trapezoidal rule since $\partial\lambda/\partial x^k$ is constant on each triangle $T \in J_h$. If the left hand side is computed with the same rule, we get a diagonal algebraic linear system whose solution is immediate.

5 Numerical examples

Let us begin with the flow in the domain $D = [0, 10] \times [h(x), 10]$ km^2. The true field $\mathbf{V}_{\text{true}} = u_{\text{true}}\mathbf{i} + w_{\text{true}}\mathbf{k}$ is calculated with the datum $u_{\text{true}} = 10$ ms^{-1} and $w_{\text{true}} = 0$ at the point ($x = 0, z = 10$ km), which is used to define the magnitude V_0 of the flow on the abstract complex plane ζ (section 3). The initial field $\mathbf{V}^0$ is $u^0 = u_{\text{true}}$ and $w^0 = 0$.

Figure 1 shows the flow with an analytic topography $h(x) = h_0 + h_1 \cos \omega x$. The true field $\mathbf{V}_{\text{true}}$ and the adjusted field $\mathbf{V}$ denoted with red and blue arrows, respectively, and the latter is computed with $\alpha_1 = \alpha_3 = 1$. Figure 2 shows the details of the fig. 1 and we see that there is a significant difference between $\mathbf{V}_{\text{true}}$ and $\mathbf{V}$. Figure 3 shows $\mathbf{V}_{\text{true}}$ and the adjusted field $\mathbf{V}$ calculated with $\varepsilon = 1/\alpha_3^2 = 10^{-2}$, we observe that $\mathbf{V}(\varepsilon = 10^{-2})$ field is worse that the field $\mathbf{V}(\varepsilon = 1)$ of fig. 2. In contrast, figure 4 shows that the adjusted field $\mathbf{V}(\varepsilon = 10^{+2})$ has the main properties of $\mathbf{V}_{\text{true}}$ and Fig. 5 shows that $\mathbf{V}(\varepsilon = 10^{+6})$ is almost equal to the true field.

Figure 6 shows the flows $\mathbf{V}_{\text{true}}$ and $\mathbf{V}(\varepsilon = 10^{+6})$ on a topography $h(x)$ defined by smoothing real terrain elevation data from GTOPO30 [14]. Details of the same figure are shown in figure 7. Once again we observe that the adjusted field is an indeed correct approximation of the true field.

These results show that MCMs's can be used to estimate the true field if the initial field $\mathbf{V}^0$ is a good approximation of the horizontal true field. The problem is : how accurate should the initial field be to obtain a reliable adjusted field?, this problem will be studied in a forthcoming work.

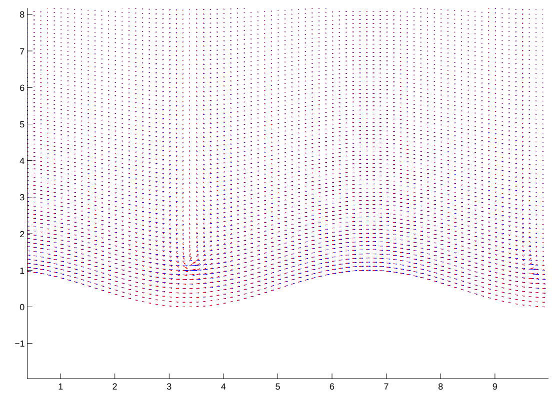

Figure 1: True and adjusted fields, $\mathbf{V}_{\text{true}}$, $\mathbf{V}(\varepsilon = 1)$

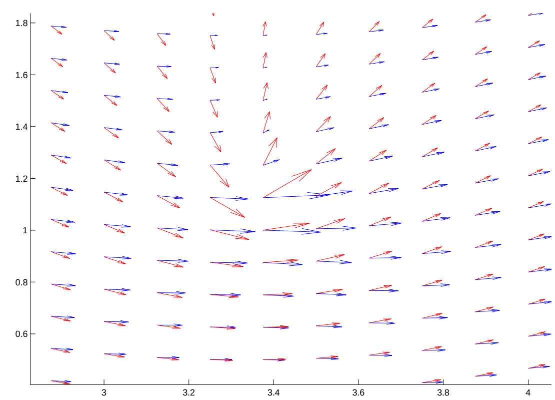

Figure 2: Details of the fields from fig. 1

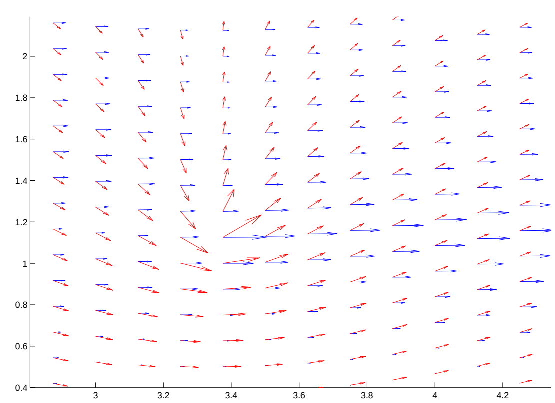

Figure 3: True and adjusted fields, $\mathbf{V}_{\text{true}}$, $\mathbf{V}(\varepsilon = 10^{-2})$

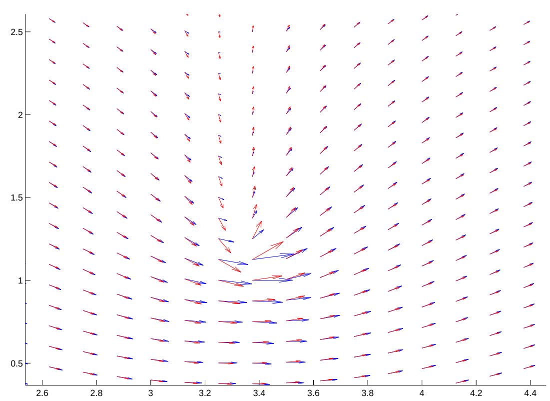

Figure 4: True and adjusted fields, $\mathbf{V}_{\text{true}}$, $\mathbf{V}(\varepsilon = 10^{+2})$

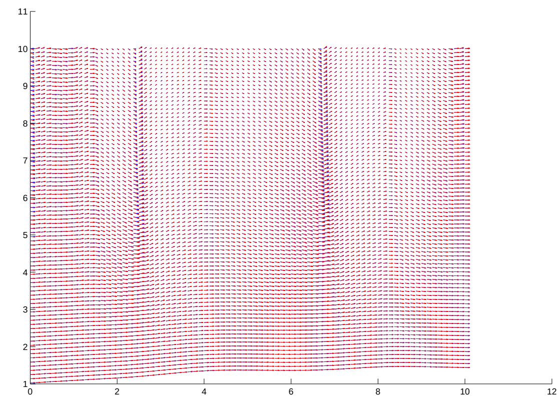

Figure 6: True and adjusted fields, $\mathbf{V}_{\text{true}}$, $\mathbf{V}(\varepsilon = 10^{+6})$ on real terrain smoothed

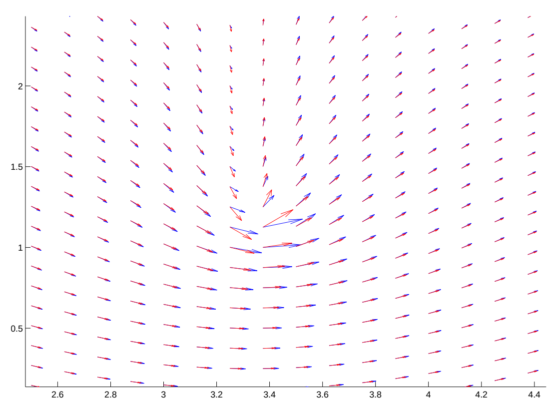

Figure 5: True and adjusted fields, $\mathbf{V}_{\text{true}}$, $\mathbf{V}(\varepsilon = 10^{+6})$

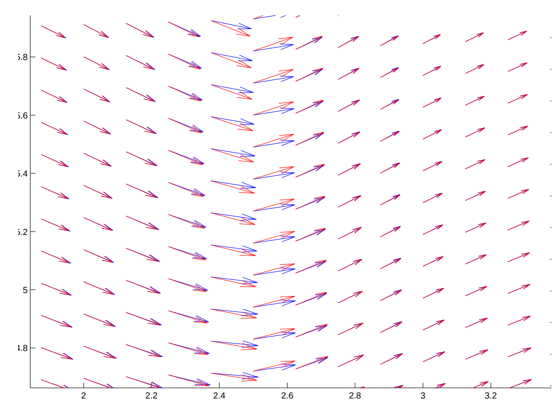

Figure 7: Details of the fields from fig. 6

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