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FORMAL PROOF OF THE EXISTENCE OF AN ATMOSPHERIC BASE-STATE AND ITS ESTIMATION

Marco A. Nuñez[1]
Universidad Autonoma Metropolitana Iztapalapa

1 Introduction

The main coordinate system used in mesoscale meteorology [1-6] is a cartesian system x^i with its origin at a point on the spherical earth with latitude ϕ_c and longitude λ_c. Let us suppose that the plane x^1x^2 is tangent to the earth at (λ_c, ϕ_c) and the axis x^3 is opposite to $\mathbf{g}$ at (λ_c, ϕ_c). In this reference system the momentum equation is

$$\frac{d\mathbf{v}}{dt} = -\rho^{-1}\nabla p + \hat{\mathbf{x}}^i g^i - 2\vec{\Omega} \times \mathbf{u} + \mathbf{f} \quad (1)$$

where p, ρ, $\mathbf{v}$ are the pressure, density and velocity vector of the particle, $\mathbf{g}$ is the gravity acceleration and $\mathbf{f}$ is a frictional force. If we assume that the earth is a sphere with radius a, then $\mathbf{g}$ is given by

$$\mathbf{g} = -g\frac{a^2}{r^3}\mathbf{R} = g^i\hat{\mathbf{x}}^i$$

where $g \equiv GMa^{-2}$, M is the earth mass, G is the gravitational constant, and $\mathbf{R}$ is the vector from the earth center to an air particle, $r = \|\mathbf{R}\|$, and the g^i's are

$$g^i = -ga^2r^{-3}(x^i + \delta_{i3}a). \quad (2)$$

The equation (1) may be referred to as the *exact momentum equation* since it has the exact components (2) of $\mathbf{g}$. In contrast, the standard mesoscale literature [1-6] uses the approximation $\mathbf{g} \sim -g\hat{\mathbf{x}}^3$ and the resulting momentum equation

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k} - 2\tilde{\mathbf{\Omega}} \times \mathbf{v} + \mathbf{f}. \quad (3)$$

Let $\mathcal{D}(L) = 2L \times 2L$ denote a rectangular region of the tangent plane x^1x^2 with center at the origin $x^i = 0$ and $|x^1|, |x^2| \le L$. In previous works [7,8] it was shown that the eq. (2) is valid on domain $\mathcal{D}(L)$ bounded by 100×100 km^2. The validity region can be estimated by the magnitude of the terms in (3). Table I yields the magnitude of the terms in the u^1-equation from (3) as reported by Atkinson [3] where we have added a column with the term $ga^2r^{-3}x^1$. We see that the term of $ga^2r^{-3}x^1$ is one order of magnitude larger than the largest term of the u^1-equation from (3) for $L = 10^2$, 10^3 km. For $L = 10$ km the magnitude of $ga^2r^{-3}x^1$ is equal to that of the Coriolis terms and 10^4 times larger than the dissipative terms. These results show that the horizontal components of $\mathbf{g}$ cannot be omitted in (3) for a region $\mathcal{D}(L)$ larger than 100×100 km^2 and should be considered for $\mathcal{D}(L)$ between 10×10 and 100×100 km^2.

TABLE I. Magnitudes in ms^{-2} of terms in the u^1-equation for flows with horizontal scale L (m), $U = 10$ ms^{-1}, $H = 10^4$ m, $f = 2\Omega\sin\phi$, $\phi = 45^o$, $g = 10$ ms^{-2}, and $x^1 = L/2$, $x^2 = x^3 = 0$, $r = \sqrt{(x^1)^2 + a^2}$, $a = 6378$ km.

	$\frac{du}{dt}$	$\frac{1}{\rho}\frac{\partial p}{\partial x^1}$	fv	fw	$\frac{\partial}{\partial x^3}K_z\frac{\partial u}{\partial x^3}$	$\frac{ga^2x^1}{r^3}$
L	$\frac{U^2}{L}$	$\frac{\Delta P}{\rho L}$	fU	$\frac{fHU}{L}$	$\frac{KU}{H^2}$	
10^6	10^{-4}	10^{-3}	10^{-3}	10^{-5}	10^{-6}	10^0
10^5	10^{-3}	10^{-2}	10^{-3}	10^{-4}	10^{-6}	10^{-1}
10^4	10^{-2}	10^{-1}	10^{-3}	10^{-3}	10^{-6}	10^{-2}

It should be noted that the scale analysis used by several authors (see, e.g., [1]) to simplify the governing equatios, starts from the approximate equation (3) but, according to Table I, the analysis should include the horizontal components of $\mathbf{g}$ for $L \ge 10$ km In ref. [7] it is

[1]Corresponding address: Departamento de Fisica, Universidad Autonoma Metropolitana Iztapalapa, A. P. 55-534, Mexico, D. F., C.P. 09340, e-mail: manp@xanum.uam.mx

shown that the linear approximations

$$g^i = -g\ x^i/a \quad (i = 1, 2)$$
$$g^3 = -g + 2\ g\ z/a \quad \text{or}\ -g$$

constitute an accurate approximation of $\mathbf{g}$ on a region $\mathcal{D}(L) \subseteq 700 \times 700$ km^2 which is large enough for several mesoscale studies.

In ref. [8] some computational mesoscale models that use the approximate equation (3) are studied. These models have been used on domains $\mathcal{D}(L)$ larger than 100×100 km^2 for the analysis of data provided by meteorological networks but no correction of (3) has been reported. This suggests that: (i) the number of data is not sufficient to see the error in (3) generated by the omission of the horizontal components of the exact gravity acceleration $\mathbf{g}$, and (ii) the results given by computational models that use equations like (3), should be reanalyzed by solving the exact momentum equations. Almost all the mesoscale literature uses (3). This includes the recent books of Pielke [1] and Jacobson [2]. Among the mesoscale models that use (3) we have the models ARPS [9], HOTMAC [10] and RAMS [11]. The factor g appears in the horizontal momentum equations of these models but, as is shown in [8], the presence of g is due to the use of terrain-following coordinates and the hydrostatic approximation.

Map projections have been used in atmospheric modeling with the aim of including the earth sphericity [6,12]. This is equivalent to rewrite the equations of motion in terms of a curvilinear coordinate system which is a legitime coordinate system [7,8]. However, models such as HOTMAC [10] and ARPS [9] use map projections only to define the topography with the data from a digital elevation model. In ref. [13] it is shown that this procedure is valid in a small region of the xy plane, which is estimated as 60×60 km^2. A careful deduction of the equations in map-projection coordinates is given in [8] where it is shown that the RAMS [11] model uses approximate momentum equations in projection coordinates because such equations are obtained from the approximated equation (3). The MM5 model [14] uses projection coordinates but, unfortunately, different versions of the model solve different governing equations [8].

The use of the approximate momentum equation (3) has motivated, in part, the following decomposition

$$\psi(\mathbf{r}, t) = \psi_0(z) + \bar{\psi}(\mathbf{r}, t) \tag{4}$$

for the dependent meteorological variables, where ψ_0 is a reference value that depends only of the height z with respect to the tangent-plane xy. Then the governing equations are used to calculate the "deviations" $\bar{\psi}(\mathbf{r}, t)$ from a known reference state $\psi_0(z)$ [1-4]. This approach is correct only if the reference value $\psi_0(z)$ is close to the total field $\psi(\mathbf{r}, t)$. However, in this work it is shown that the decomposition (4) is valid on a small region $\mathcal{D}(L)$, as occurs with the momentum equation (3). In section 2 we use the correct momentum equation (1) to prove that each meteorological field has the decomposition

$$\psi(\mathbf{r}, t) = \psi^{(0)}(z_s, t) + \sum_{k=1} \psi^{(k)}(\mathbf{r}, t)\ \mu^k \tag{5}$$

where z_s is the height with respect to an spherical earth model and μ is a suitable parameter. Analytic solutions of the continuity equation $\nabla \cdot \mathbf{V} = 0$ [17] are used in section 3 to show that the terms $\psi^{(0)}(z_s, t)$ can be estimated by means of a spatial average. The same analytic flows are used to prove that the decomposition (4) is valid on a region $\mathcal{D}(L)$ bounded by 100×100 km^2, a result that is consistent with the small validity region of (3).

The proof (5) is based on the replacement of the momentum equation (1) by an ordinary differential equation. In section 4 it is shown that such a differential equation provides a very simple diagnostic scheme to compute compute the pressure field from the velocity alone, without the knowledge of any other thermodynamic variable. The simplicity of the scheme is shown by means of an analytic example. In contrast, the standard diagnostic schemes (see, e.g., [1]) pose the formidable problem of solving a *non*linear partial differential equation.

2 Proof of decomposition (5)

Let λ, ϕ, r be the geographic coordinates defined with respect to a cartesian system XYZ fixed to the earth and its origin is at the earth's center,

$$X = r\cos\phi\cos\lambda \quad Y = r\cos\phi\sin\lambda \quad Z = r\sin\phi.$$

In terms of the curvilinear coordinates

$$x_s = (\lambda - \lambda_c)a\cos\phi \qquad y_s = (\phi - \phi_c)a \qquad z_s = r - a.$$

the momentum equations are (frictional terms are neglected)

$$\frac{du_s}{dt} - \frac{u_s v_s}{r}\tan\phi + \frac{u_s w_s}{r} - 2\Omega v_s \sin\phi + 2\Omega w_s\cos\phi = -\frac{a\cos\phi_c}{r\cos\phi}\frac{1}{\rho_s}\frac{\partial p_s}{\partial x_s} \tag{6}$$

$$\frac{dv_s}{dt} + \frac{u_s^2}{r}\tan\phi + \frac{v_s w_s}{r} + 2\Omega u_s \sin\phi + \Omega^2 r\cos\phi\sin\phi = -\frac{a}{r}\frac{1}{\rho_s}\frac{\partial p_s}{\partial y_s}$$

$$\frac{dw_s}{dt} - \frac{u_s^2 + v_s^2}{r} - 2\Omega u_s\cos\phi - \Omega^2 r\cos^2\phi = -\frac{1}{\rho_s}\frac{\partial p_s}{\partial z_s} - g\frac{a^2}{r^2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_s\frac{a\cos\phi_c}{r\cos\phi}\frac{\partial}{\partial x_s} + v_s\frac{a}{r}\frac{\partial}{\partial y_s} + w_s\frac{\partial}{\partial z_s}.$$

The key of our argument to prove the decomposition (5) consists in replacing these equations by the Ordinary Differential Equation (ODE) associated to a pressure-constant curve (which is usually called *isobar*). To this end let us introduce quasi-polar coordinates ξ_s, θ_s defined by

$$x_s = \xi_s\cos\theta_s \qquad y_s = \xi_s\sin\theta_s.$$

The isobar defined by the intersection of the surface $\theta_s = \theta_s^0 (= cte.)$ and a pressure-constant surface,

$$p_s(x_s, y_s, z_s, t) = p_s^0 = cte.,$$

has parametric equations

$$x_s = \xi_s\cos\theta_s \quad y_s = \xi_s\sin\theta_s \quad z_s = f(\xi_s, \theta_s, t),$$

where θ_s and t are taken as *constant* parameters, which obviously satisfy

$$p_s(x_s = \xi_s\cos\theta_s, y_s = \xi_s\sin\theta_s, z_s = f_s(\xi_s, \theta_s, t),\ t\ ) = p_s^0$$

Differentiation with respect to ξ_s yields

$$\frac{df}{d\xi_s} = -\cos\theta_s\frac{\partial_{xs}p_s}{\partial_{zs}p_s} - \sin\theta_s\frac{\partial_{ys}p_s}{\partial_{zs}p_s} \tag{7}$$

where the ratios

$$\frac{\partial_{xs}p_s}{\partial_{zs}p_s} \qquad \frac{\partial_{ys}p_s}{\partial_{zs}p_s}$$

are obtained from the momentum equations (6). This defines an ODE for f if the velocity field and frictional forces are known. The solution of this equation with the boundary condition

$$f_s = z_0 \quad \text{at } \xi_s = 0 \tag{8}$$

yields an isobar that passes through the point ($x_s = y_s = 0$, $z_s = z_{s0}$). The equation (7) is worthy because it does not contain the density ρ.

Let L_x, L_y, H, t_0, U_0, V_0, W_0 be the characteristic values of x_s, y_s, z_s, t, u_s, v_s y w_s, respectively, with $L_x = L_y \equiv L$, $U_0 = V_0$ and $t_0 \equiv L/U_0$. The dimensionless variables are

$$\begin{aligned} \bar{x}_s &= x_s/L \quad \bar{y}_s = y_s/L \quad \bar{z}_s = z_s/H \quad \bar{t} = t/t_0,\\ \bar{r} &= r/a, \quad \bar{u} = u/U_0 \quad \bar{v} = v/U_0 \quad \bar{w} = w/W_0,\\ \bar{f}_s &= f_s/H, \quad \bar{\xi}_s = \xi_s/L \end{aligned}$$

and let us define the dimensionless parameters

$$\varepsilon = \frac{H}{L} \qquad \eta = \frac{W_0}{U_0}, \quad \delta = \frac{L}{a}, \quad \mu = \frac{U_0^2}{gL}\ .$$

Let p_r and T_r the characteristic values of pressure and temperature at the surface earth, the characteristic value ρ_r of the density is obtained from the equation of state, $p_r = \mathcal{R}T_r\rho_r$. It is generally accepted that the characteristic values observed for midlatitude large-scale synoptic systems are [15]

$$\begin{aligned} L &= 10^3 \text{ km} \qquad H = 10 \text{ km}, \quad U_0 = 10 \text{ ms}^{-1}\\ W_0 &= 10^{-2} \text{ ms}^{-1}, \ \varepsilon = 10^{-2} \qquad \eta = 10^{-3},\\ \delta &= 10^{-1}, \qquad \mu = 10^{-5}. \end{aligned}$$

In terms of dimensionless variables the momentum equations (6) take the form

$$\frac{d\bar{u}}{d\bar{t}} - \delta\frac{\bar{u}\bar{v}}{\bar{r}}\tan\phi + \delta\eta\frac{\bar{u}\bar{w}}{\bar{r}} + \frac{2\Omega L}{U_0}(\eta\bar{w}\cos\phi - \bar{v}\sin\phi) = -\frac{p_r/\rho_r}{U_0^2}\frac{\cos\phi_c}{\cos\phi}\frac{1}{\bar{r}\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{x}_s}$$

$$\frac{d\bar{v}}{d\bar{t}} + \delta\frac{\bar{u}^2\tan\phi}{\bar{r}} + \delta\eta\frac{\bar{v}\bar{w}}{\bar{r}} + \frac{2\Omega L}{U_0}\bar{u}\sin\phi + \frac{\Omega^2 aL}{U_0^2}\bar{r}\cos\phi\sin\phi = -\frac{p_r/\rho_r}{U_0^2}\frac{1}{\bar{r}\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{y}_s}$$

$$\varepsilon\eta\frac{d\bar{w}}{d\bar{t}} - \delta\varepsilon\frac{\bar{u}^2 + \bar{v}^2}{\bar{r}} - \frac{2\Omega L}{U_0}\varepsilon\bar{u}\cos\phi - \frac{\Omega^2 aL}{U_0^2}\varepsilon\bar{r}\cos^2\phi = -\frac{p_r/\rho_r}{U_0^2}\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{z}_s} - \frac{gH}{U_0^2\bar{r}^2}$$

where

$$\frac{d}{d\bar{t}} = \frac{\partial}{\partial\bar{t}} + \frac{\bar{u}_s}{\bar{r}}\frac{\cos\phi_c}{\cos\phi}\frac{\partial}{\partial\bar{x}_s} + \frac{\bar{v}_s}{\bar{r}}\frac{\partial}{\partial y_s} + \frac{\eta}{\varepsilon}\bar{w}_s\frac{\partial}{\partial\bar{z}_s}\ .$$

Let p_{xs} be defined by

$$p_{xs} \equiv \frac{d\bar{u}}{d\bar{t}} - \delta\frac{\bar{u}\bar{v}}{\bar{r}}\tan\phi + \delta\eta\frac{\bar{u}\bar{w}}{\bar{r}} + \frac{2\Omega L}{U_0}(\eta\bar{w}\cos\phi - \bar{v}\sin\phi)\ ,$$

and let us write the left hand side of the $\bar{w}-$equation as follows

$$\begin{aligned}
&\varepsilon\eta\frac{d\bar{w}}{d\bar{t}} - \delta\varepsilon\frac{\bar{u}^2+\bar{v}^2}{\bar{r}} - \frac{2\Omega L}{U_0}\varepsilon\bar{u}\cos \\
&\qquad - \frac{\Omega^2 aL}{U_0^2}\varepsilon\bar{r}\cos^2\phi + \frac{gH}{U_0^2}\bar{r}^{-2} \\
&= \left\{\left(\eta\frac{d\bar{w}}{d\bar{t}} - \delta\frac{\bar{u}^2+\bar{v}^2}{\bar{r}} - \frac{2\Omega L}{U_0}\bar{u}\cos\phi\right.\right. \\
&\qquad \left.\left. -\frac{\Omega^2 aL}{U_0^2}\bar{r}\cos^2\phi\right)\varepsilon\frac{U_0^2\bar{r}^2}{gH} + 1\right\}\frac{gH}{U_0^2}\bar{r}^{-2} \\
&= \left\{\left(\eta\frac{d\bar{w}}{d\bar{t}} - \delta\frac{\bar{u}^2+\bar{v}^2}{\bar{r}} - \frac{2\Omega L}{U_0}\bar{u}\cos\phi\right.\right. \\
&\qquad \left.\left. -\frac{\Omega^2 aL}{U_0^2}\bar{r}\cos^2\phi\right)\mu\bar{r}^2 + 1\right\}\frac{\bar{r}^{-2}\varepsilon}{\mu}\bar{r}^{-2} \\
&= (1+\mu p_{zs})\frac{\bar{r}^{-2}\varepsilon}{\mu}
\end{aligned}$$

where we define

$$p_{zs} \equiv \left(\eta\frac{d\bar{w}}{d\bar{t}} - \delta\frac{\bar{u}^2+\bar{v}^2}{\bar{r}} - \frac{2\Omega L}{U_0}\bar{u}\cos\phi - \frac{\Omega^2 aL}{U_0^2}\bar{r}\cos^2\phi\right)\bar{r}^2 \ .$$

Hence we get

$$\frac{\frac{\cos\phi_c}{\cos\phi}\bar{r}^{-1}\frac{\partial\bar{p}}{\partial\bar{x}_s}}{\frac{\partial\bar{p}}{\partial\bar{z}_s}} = \frac{\mu}{\bar{r}^{-2}\varepsilon}\frac{p_{xs}}{1+\mu p_{zs}} \ . \tag{9}$$

In a similar way we have

$$\frac{\bar{r}^{-1}\frac{\partial\bar{p}}{\partial\bar{y}_s}}{\frac{\partial\bar{p}}{\partial\bar{z}_s}} = \frac{\mu}{\bar{r}^{-2}\varepsilon}\frac{p_{ys}}{1+\mu p_{zs}} \tag{10}$$

with

$$\begin{aligned}
p_{ys} \equiv\ & \frac{d\bar{v}}{d\bar{t}} + \delta\frac{\bar{u}^2}{\bar{r}}\tan\phi + \delta\eta\frac{\bar{v}\bar{w}}{\bar{r}} \\
& + \frac{2\Omega L}{U_0}\bar{u}\sin\phi + \frac{\Omega^2 aL}{U_0^2}\bar{r}\cos\phi\sin\phi
\end{aligned}$$

and the ODE (7) takes the form

$$\frac{d\ \bar{f}_s}{d\ \bar{\xi}_s} = -\frac{\mu}{\varepsilon}\left[\cos\theta_s\frac{\cos\phi}{\cos\phi_c}p_{xs} + \sin\theta_s p_{ys}\right]\frac{\bar{r}^3}{1+\mu p_{zs}} \tag{11}$$

with $\bar{r} = r/a \sim 1$, and $p_{xs} \sim 10$, $p_{ys} \sim 10^2$, $\mu p_{zs} \sim 10^{-3}$. The eq. (12) has the form

$$\frac{d\bar{f}_s}{d\bar{\xi}_s} = \mu F(\bar{\xi}_s, \bar{f}_s, \bar{t}, \theta_s, \mu) \ \text{ with } \bar{f}_s = \bar{z}_0 \text{ at } \bar{\xi}_s = 0. \tag{12}$$

We can invoke the next result.

Theorem [16, p. 213]. Let $X = (x_1, x_2, ..., x_n)$. If $F(X, z, \mu)$ is analytic in x_1, x_2,..., x_n and μ, and continuous in ξ, then the system

$$\frac{dX}{d\xi} = F(X, \xi, \mu)$$

under the condition $X(\xi = 0, \mu) = 0$, solution has unique solution of the form

$$X = \sum_{k=0}^{\infty} X_k(\xi)\mu^k \ .$$

where $X_k(\xi)$ is a vector function of ξ.

In our case the right side of (11) is an *analytic* function of μ in a vicinity of $\mu = 0$ and we can suppose that $F(\bar{\xi}_s, \bar{f}_s, \bar{t}, \theta_s, \mu)$ is analytic in $\bar{f}_s$ and continuous in $\bar{\xi}_s$, then there exists a solution of the form

$$\bar{f}_s = \sum_{k=0}^{\infty} \bar{f}^{(k)}(\bar{\xi}_s, \bar{f}_s, \bar{t}, \theta_s)\ \mu^k \ , \tag{13}$$

and it is easy to see that the coefficients $\bar{f}_s^{(k)}$ satisfy the boundary conditions

$$\bar{f}^{(0)} = \bar{z}_0 \ , \qquad \bar{f}^{(k)} = 0 \ \text{ for } k \geq 1.$$

Replacing the series (13) into (12) we get the zero-order solution

$$f^{(0)}(\xi_s, t, \theta_s) = z_0.$$

This means that the zero-order isobar that passes through ($x_s = y_s = 0$, $z_s = z_{s0}$) depends only of z_s and, consequently, the pressure field has the form

$$p(\mathbf{r}, t) = p^{(0)}(z_s, t) + \sum_{k=1} p^{(k)}(\mathbf{r}, t)\ \mu^k. \tag{14}$$

This rigorous result is worthy because it is completely independent of the distribution of temperature and density. The substitution of (14) into the equation of state

$$p(\mathbf{r}, t, \mu) = \mathcal{R}\ T(\mathbf{r}, t, \mu)\ \rho(\mathbf{r}, t, \mu),$$

yields a functional equation for $T(\mathbf{r}, t, \mu)$ and $\rho(\mathbf{r}, t, \mu)$ which has the formal solution

$$T(\mathbf{r}, t) = T^{(0)} + \sum_{k=1} T^{(k)}(\mathbf{r}, t)\ \mu^k \tag{15}$$

$$\rho(\mathbf{r}, t) = \rho^{(k)} + \sum_{k=1} \rho^{(k)}(\mathbf{r}, t)\ \mu^k \ . \tag{16}$$

where it is easy to see that the zero-order terms only depend of z_s and t,

$$\begin{aligned} T^{(0)} &= T^{(0)}(z_s,t) \\ \rho^{(0)} &= \rho^{(0)}(z_s,t). \end{aligned}$$

In other words, *there exists a thermodynamic reference state that depends only of* z_s and t. The substitution of the series (14-16) into the momentum equation or the mass-conservation equation leads to the series

$$\mathbf{V}(\mathbf{r},t) = \mathbf{V}^{(0)}(z_s,t) + \sum_{k=1} \mathbf{V}^{(k)}(\mathbf{r},t)\ \mu^k. \qquad (17)$$

Thus we conclude that each dependent meteorological variable ψ has the series

$$\psi(\mathbf{r},t) = \psi^{(0)}(z_s,t) + \sum_{k=1} \psi^{(k)}(\mathbf{r},t)\ \mu^k\ . \qquad (18)$$

The standard decomposition (4) is an approximation of (18) because the relevant coordinate is the height z_s with respect to the earth rather than the height z with respect to the tangent-plane xy.

3 Estimation of the reference value $\psi^{(0)}(z_s,t)$

In this section we consider the estimation of the term $\psi^{(0)}(z_s,t)$ in (18) by means of the spatial average

$$\hat{\psi}(z_s) = \frac{1}{L_s}\int_{-L_s/2}^{+L_s/2} \psi(x_s,z_s)\ dx_s \qquad (19)$$

for the case of a stationary bidimensional flow on the $xz-$plane. The flow in question $\mathbf{V} = u\mathbf{i} + w\mathbf{k}$ is an analytic solution of the so-called [1] shallow continuity equation

$$\nabla\cdot\mathbf{V}(\mathbf{r}) = 0 \qquad (20)$$

with the boundary condition

$$\mathbf{V}\cdot\mathbf{n} = 0 \quad \text{on} \quad z = h(x) \qquad (21)$$

where $h(x)$ is the terrain elevation on the point $(x, y = 0, z = 0)$.

3.1 Calculation of V

Let us describe briefly the method used to obtain $\mathbf{V}$ [17]. To begin consider an abstract complex plane with variable

$$\zeta = \bar{x} + i\ \bar{z}\ .$$

In this plane we consider a uniform flow

$$\bar{V} = V_0 \qquad (\bar{u} = V_0,\ \bar{w} = 0)$$

obtained from the potential

$$\bar{\phi} = V_0\bar{x}.$$

The physical space can be seen as the complex plane associated to the variable

$$\xi = x + i\ z\ .$$

Suppose that $h(\zeta)$ is an analytic function of ζ and let h_1 and h_2 be the real and imaginary parts of $h(\zeta)$,

$$h(\zeta) = h_1(\bar{x},\ \bar{z}) + ih_2(\bar{x},\ \bar{z})\ .$$

Then the function

$$G(\zeta) = \zeta + ih(\zeta)$$

is also an analytic function of ζ and defines the transformation equations

$$\begin{aligned} x &= x(\bar{x},\ \bar{z}) = \bar{x} - h_2(\bar{x},\ \bar{z}) \\ z &= z(\bar{x},\ \bar{z}) = \bar{z} + h_1(\bar{x},\ \bar{z}). \end{aligned}$$

It is clear that the imagine of the real axis $\bar{z} = 0$ in the $\zeta-$plane under these transformation equations is the curve defined by the the topography,

$$\{(x,h(x)\} = G[\{\bar{x},\bar{z} = 0\}];$$

that is, we have

$$x = \bar{x}, \qquad z = h(\bar{x})\ .$$

Since the real axis $\bar{z} = 0$ is a stream line of the flow $\bar{V}$, the curve $z = h(\bar{x})$ is a stream line of the flow V that is the image of $\bar{V}$ under the transformation G. The components of the flow $\mathbf{V} = u\mathbf{i} + w\mathbf{k}$ are

$$u = \frac{V_0}{J}\frac{\partial\ z(\bar{x},\ \bar{z})}{\partial\ \bar{z}} \qquad w = -\frac{V_0}{J}\frac{\partial\ x(\bar{x},\ \bar{z})}{\partial\ \bar{z}}$$

where

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial \bar{z}} \\ \frac{\partial z}{\partial \bar{x}} & \frac{\partial z}{\partial \bar{z}} \end{pmatrix}$$

and, since $G(\zeta)$ is analytic, the Cauchy-Riemann equations hold,

$$\frac{\partial x}{\partial \bar{x}} = \frac{\partial z}{\partial \bar{z}} \qquad \frac{\partial x}{\partial \bar{z}} = -\frac{\partial z}{\partial \bar{x}}\ .$$

Inherent problems of the map conforming do not permit us the direct use of $h(x)$. These problems are solved by a simpler representation of $h(x)$, namely, a *natural spline* $S(x)$ which is defined as follows. Let $\{x_k\}_{k=0}^{n}$ be a set of points where the terrain height $h(x_k)$ is known, then : (i) $S(x)$ satisfies

$$S(x_k) = h(x_k) \text{ for } k = 0, ..., n,$$

(ii) $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$,

$$\begin{aligned} S(x) &= a_k + b_k(x - x_k) + c_k(x - x_k)^2 \\ &\quad + d_k(x - x_k)^3 \text{ for } x \in [x_k, x_{k+1}], \end{aligned}$$

(iii) $S(x)$ and its derivatives $S'(x)$, $S''(x)$ are continuous on $[x_0, x_n]$ and $S''(x)$ satisfies

$$S''(x_0) = S''(x_n) = 0.$$

There is a unique natural spline associated to an analytic function $h(x)$ on the interval $[x_0, x_n]$. Since $S(x)$ is a cubic polynomial on each interval $[x_k, x_{k+1}]$, we can compute the flow

$$\begin{aligned} u &= \frac{V_0}{J}\frac{\partial z^{(k)}}{\partial \bar{z}} \\ w &= -\frac{V_0}{J}\frac{\partial x^{(k)}}{\partial \bar{z}} \quad \text{for } x \in [x_k, x_{k+1}], \end{aligned}$$

where

$$\begin{aligned} x^{(k)} &= \bar{x} - S_2(\bar{x},\ \bar{z}) \\ z^{(k)} &= \bar{z} + S_1(\bar{x},\ \bar{z}) \end{aligned}$$

and

$$S(\zeta = \bar{x} + i\bar{z}) = S_1(\bar{x},\ \bar{z}) + i\ S_2(\bar{x},\ \bar{z}).$$

The continuity of $S(x)$, $S'(x)$ and $S''(x)$ guarantees that the field $\mathbf{V} = u\mathbf{i} + w\mathbf{k}$, its first derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial z},$$

and $\nabla \cdot \mathbf{V}$ are continuous on the interval $[x_0, x_n]$. This together with the fact that u, w satisfy the continuity equation (20) and the boundary condition (21) on each interval $[x_k, x_{k+1}]$, implies that the field $\mathbf{V}$ satisfies the same equations on the whole interval $[x_0, x_n]$. Figure 1 shows the region and some points where $\mathbf{V}$ is computed. Data from the data base GTOPO30 [18] were used to estimate $h(x)$ by means of a spline $S(x)$. The field $\mathbf{V} = u\mathbf{i} + w\mathbf{k}$ is calculated with the datum $u = 10$ ms^{-1} and $w = 0$ at the point $(x = 0, z = 10$ km).

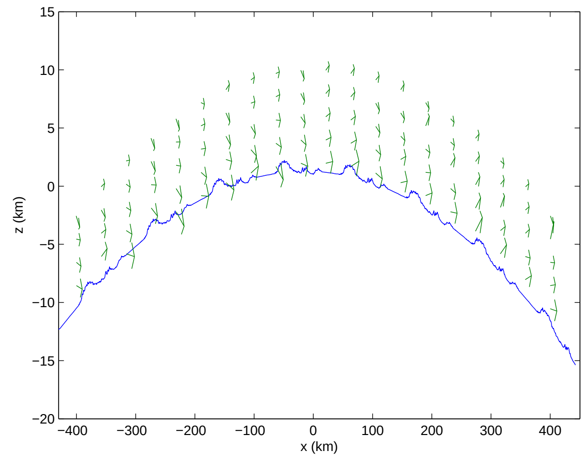

Figure 1: Topography and sketch of the field $\mathbf{V}$

3.2 Estimation of $p^{(0)}(z_s, t)$

Let us consider the estimation of the term $p^{(0)}(z_s, t)$ corresponding to the decomposition (19)

$$p(x_s, z_s) = p^{(0)}(z_s) + \sum_{k=1}^{\infty} p^{(k)}(\mathbf{r})\ \mu^k\ , \tag{22}$$

for the pressure p, by means of the spatial average

$$\hat{p}(z_s) = \frac{1}{L_s}\int_{-L_s/2}^{+L_s/2} p(x_s, z_s)\ dx_s \tag{23}$$

The Bernoulli equation is used to obtain the pressure field at a point $\mathbf{r} = (x, y, z)$,

$$\frac{p(\mathbf{r})}{\rho_0} = C_0 - \frac{1}{2}V^2(\mathbf{r}) + \phi_g(\mathbf{r})$$

where $V^2 = u^2 + w^2$, $\rho_0 = 1$ kg/m^3, $\phi_g(\mathbf{r}) = -ga^2/(z_s + a)$ is the gravitational potential, $a = 6376.98$ km is the earth radius and $C_0 = -62428\times10^4$ m^2/s^2 is calculated with $p(\mathbf{r}) = 324.84$ mb, $u = 10.0$ m^2/s^2, $w = 0.13$ m^2/s^2 $\phi_g(\mathbf{r}) = -62460\times10^4$ m^2/s^2 at $(x = 0,\ y = 0,\ z_s = 10$ km). The relative error of $\hat{p}(z_s)$ with respect to the exact value $p(x, z_s)$,

$$\Delta\hat{p}(z_s) = \left(1 - \frac{\hat{p}(z_s)}{p(x_s, z_s)}\right) \times 100\ ,$$

is a suitable measure of the accuracy of $\hat{p}$ if it is seen as an approximation of $p(x_s, z_s)$ and the reference state $p^{(0)}(z_s)$.

Table II shows the results for the pressure. Surprisingly, we see that $\hat{p}(z_s)$ has a relative error $|\Delta\hat{p}(z_s)|$ that is lower than 0.1 % for L_s from 50 to 800 km, so that for practical purposes $\hat{p}(z_s)$ is almost equal to $p(x, z_s)$ and, consequently, the field $p(\mathbf{r})$ has the decomposition

$$p(x_s, z_s) = \hat{p}(z_s) + \delta\hat{p}(x_s, z_s)$$

with $|\delta\hat{p}/p| < 10^{-1}$ %. In ref. [17] we report analytic solutions of the so-called deep continuity equation $\nabla \cdot \rho_0(\mathbf{r})\mathbf{U}(\mathbf{r}) = 0$. The use of the field $\mathbf{U}$ leads basically to the same numerical values of Table II. These results support the correctness of the decomposition (5) and the estimation of $\psi^{(0)}(z_s, t)$ by means of a spatial average like $\hat{\psi}(z_s)$ [eq. (19)].

TABLE II. Values of $\hat{p}$, $\min\{\Delta\hat{p}\}$ and $\max\{\Delta\hat{p}\}$ for the pressure (in mb) at $z_s = 2$, 10 km.

L_s	$\min\{\Delta\hat{p}\}$	$\max\{\Delta\hat{p}\}$	$\hat{p}$
		$z_s = 2$ km	
800 km	−.1	.1	1106.64
400 km	−.1	.1	1106.60
100 km	−.1	.0	1106.65
50 km	.0	.0	1106.67
L_s	$\min\{\Delta\hat{p}\}$	$\max\{\Delta\hat{p}\}$	$\hat{p}$
		$z_s = 10$ km	
800 km	.0	.0	324.85
400 km	.0	.0	324.85
100 km	.0	.0	324.85
50 km	.0	.0	324.86

3.3 Validity region of decomposition (4)

Let us consider the standard mesoscale decomposition

$$\psi(\mathbf{r}) = \psi_0(z) + \bar{\psi}(\mathbf{r})$$

where the reference value $\psi_0(z)$ is estimated by means of the spatial average (see , e.g., [1])

$$\breve{\psi}(z) = \frac{1}{L}\int_{-L/2}^{+L/2} \psi(x, z)\ dx. \tag{24}$$

The relative error of $\breve{\psi}(z)$ with respect to the exact value $\psi(x, z)$,

$$\Delta\breve{\psi}(z) = \left(1 - \frac{\breve{\psi}(z)}{\psi(x, z)}\right) \times 100\ ,$$

is a suitable measure of accuracy of $\breve{\psi}$ if it is seen as an approximation of $\psi(x, z)$ and the reference state $\psi^{(0)}(z)$. Table III reports results for the pressure obtained from the same flow of Table II and the Bernoulli equation. We see that the relative error $|\Delta\breve{p}|$ is lower than 5% for $L \leq 100$ km. The use of analytic solutions of the so-called deep continuity equation $\nabla \cdot \rho_0(\mathbf{r})\mathbf{U}(\mathbf{r}) = 0$ yields basically to the same numerical values of Table III. If we consider that the average value $\breve{p}(z)$ is obtained from an exact field $p(x, z)$ whereas in practical situations $\breve{p}(z)$ is estimated from the data of an observational network and, consequently, is accuracy with respect to the true value $\breve{p}(z)$, may be poor, then the validity of the decomposition (4) may be significantly lower than 100×100 km^2.

TABLE III. Values of $\breve{p}$, $\min\{\Delta\breve{p}\}$ and $\max\{\Delta\breve{p}\}$ for the pressure (in mb) at $z = 2$, 10 km.

L	$\min\{\Delta\breve{p}\}$	$\max\{\Delta\breve{p}\}$	$\breve{p}$
		$z_s = 2$ km	
400 km	−20.2	10.4	1002.63
200 km	−4.7	2.4	1080.71
150 km	−2.6	1.3	1092.42
100 km	−1.2	0.5	1100.87
50 km	−0.3	0.2	1105.02
L	$\min\{\Delta\breve{p}\}$	$\max\{\Delta\breve{p}\}$	$\breve{p}$
		$z_s = 10$ km	
400 km	−91.5	45.8	222.82
200 km	−17.0	8.5	299.28
150 km	−9.2	4.6	310.44
100 km	−4.0	2.0	318.42
50 km	−1.0	0.5	323.20

4 Estimation of thermodynamic variables

Doppler Radars can provide detailed wind data so that three-dimensional kinematics of small-scale convective systems can be constructed. The radars cannot, however, measure directly the three-dimensional structure of pressure, temperature and density. The main approach to estimate the pressure consists in replacing the momentum equation by a Poisson's partial differential equation for the pressure [1],

$$\nabla^2 p = \nabla \cdot \rho\left(-\frac{d\mathbf{v}}{dt} + \hat{\mathbf{x}}^i g^i - 2\vec{\Omega} \times \mathbf{u} + \mathbf{f}\right). \tag{25}$$

where we consider the exact equation (1) instead of the approximate one (3). This equation can be solved with the Neumann Boundary Conditions (NBCs')

$$\frac{\partial p}{\partial n} = \hat{\mathbf{n}} \cdot \nabla p = \hat{\mathbf{n}} \cdot \rho\left(-\frac{d\mathbf{v}}{dt} + \hat{\mathbf{x}}^i g^i - 2\vec{\Omega} \times \mathbf{u} + \mathbf{f}\right) \tag{26}$$

obtained from the same momentum equation. This scheme has the following disadvantages [1]: (i) The solution of (25,26) is not unique. (ii) The computing time is increased by the numerical solution of (25,26). (iii) The numerical differentiation magnifies errors. (iv) The equation has an additional unknown, the density, which is approximated by its "hydrostatic" value. These problems are *easily* eliminated by means of the Ordinary Differential Equation (ODE) (7), which has a solution uniquely determined by the boundary condition (8). Additionally, *our scheme has the advantage that the* sole *momentum equation allows us to compute all the pressure constant surfaces because the density disappears in (7).*

In order to show the simplicity of the ODE (7) we consider an analytic calculation of thermodynamic variables with respect to the tangent-plane coordinate system xyz and a uniform velocity field

$$\mathbf{V} = \mathbf{i}U + \mathbf{j}V.$$

In ref. [7] it is shown that the components of $\mathbf{g}$ can be approximated by

$$g^1 = -gx/a, \quad g^1 = -gy/a, \quad g^3 = -g$$

on a domain $\mathcal{D}(L) \sim 700 \times 700$ km^2. As in section 2, polar coordinates are used,

$$x = \xi \cos\theta \qquad y = \xi \sin\theta.$$

Let $z = f(\xi, \theta, z_0)$ denote the isobar defined by the intersection of the plane θ =cte. and the pressure-constant surface at $(x = y = 0, z_0)$, then the momentum equation (1) leads to an equation of the form

$$\frac{df}{d\xi} = \tilde{A}\,\xi + \tilde{B}$$

whose solution with $f = z_0$ at $\xi = 0$ is

$$f(\xi, \theta, z_0) = z_0 + \alpha x - \beta y + \gamma(x^2 + y^2)$$

where α, β, γ are constants. This is exactly the cartesian equation of the pressure-constants surface that passes by $(x = y = 0, z_0)$. This result *cannot* be obtained from equations (25,26) if the density distribution ρ is *unknown*. Let $p_0[z]$ be the pressure distribution on the z-axis provided by a radiosonde. It is easy to see that this *sole* information allows us to obtain the pressure distribution at an *arbitrary* point (x, y, z)!, namely,

$$p(x, y, z) = p_0\left[z - \alpha x + \beta y - \gamma(x^2 + y^2)\right].$$

Once again, this result *cannot* be obtained from (25,26) if the density ρ is unknown. In order to compute ρ and T, let us consider the case of an adiabatic atmosphere. Then the well-known thermodynamic Poisson's formulas

$$\begin{aligned} T &= T_0(p/p_0)^{R/C_p} \\ \rho &= \mathcal{R}T/p \end{aligned}$$

yield T and ρ in terms of p. In contrast, if these equations are used to eliminate ρ from (25,26), we have the formidable problem of solving a *nonlinear* partial differential equation to compute p. A deeper study about the equations (25,26), our diagnostic scheme and additional examples, will be given in a forthcoming work [19].

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