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THE BIAS IN MOMENT ESTIMATORS FOR PARAMETERS OF DROP-SIZE DISTRIBUTION FUNCTIONS: SAMPLING FROM GAMMA DISTRIBUTIONS

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Abstract

This paper complements an earlier one, showing that the moment estimators frequently used to estimate parameters for drop-size-distribution (DSD) functions being "fitted" to observed raindrop size distributions are biased. The "fitted" functions often do not represent well either the raindrop samples or the underlying populations from which the samples were taken. Monte Carlo simulations of the process of sampling from known gamma DSDs (of which the exponential DSD treated in the earlier paper is a special case), followed by application of a variety of moment estimators, demonstrate this bias. Skewness in the sampling distributions of the DSD moments is the root cause of this bias; this skewness decreases as the shape parameter of the population gamma DSD increases, but increases with the order of the moment. As a result, the bias is stronger when higher-order moments are used in the procedures. Correlations of the sample moments with the size of the largest drop in a sample (D_{max}) are weaker than for the case of sampling from an exponential DSD, and correlations of the estimated parameters with D_{max} noted in that case are not present here. However, spurious correlations between the estimated parameters remain. These things can lead to erroneous inferences about characteristics of the raindrop populations being sampled. The bias, and the correlations, diminish as the sample size increases, so that with large samples the moment estimators may become sufficiently accurate for many purposes.

1. INTRODUCTION

Investigators frequently acquire observations of raindrop sizes and seek to describe the drop-size distributions (DSDs) of the underlying populations from which the samples were taken by analytical expressions, the exponential or gamma function being most common. While moment methods to estimate parameters for the DSD functions have become more or less traditional, Haddad *et al.* (1996, 1997) pointed out that such methods are biased. Statisticians (e.g. Robertson and Fryer, 1970) and hydrologists (e.g. Wallis *et al*. 1974) have long been aware of this bias, and Smith and Kliche (2005) provided examples of the bias for the case of sampling from an exponential raindrop DSD. Though the intuitive appeal of the moment approach seems almost irresistible, and the associated mathematical manipulations lend a convincing aura, the methods are indeed biased – in the statistical sense that the expected values of the "fitted" parameters differ from the parameters of the underlying raindrop populations. That can lead to erroneous inferences about the characteristics of the DSDs being sampled.

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The bias in the moment methods can be demonstrated by testing their ability to recover parameters of known DSDs from which samples are taken. This must be done by computer simulation, as the DSDs in nature are inherently unknown. The case of sampling from exponential DSDs was treated in Smith and Kliche (2005), and the present paper gives results for samples taken from hypothetical gamma DSDs and "fitted" with various moment-based procedures. The simulations use a Monte Carlo simulation procedure similar to that described in Smith and Kliche and outlined below.

2. SIMULATION OF RAINDROP SAMPLING

We seek to determine the sampling distributions of DSD parameters "fitted" to raindrop observations by moment methods. This is done by simulating repetitive sampling from a specified population DSD, gamma in form for the present paper. A gamma DSD is usually expressed in the form

$$n\left(D\right)=n_1 D^{\mu}\exp\left(-\lambda D\right) \quad (1)$$

where $n(D)$ is the number concentration of drops of diameter D, per unit size interval, and n_1, μ and λ are concentration, distribution shape, and size (scale) parameters, respectively, of the DSD. It is customary to portray such DSDs on semilogarithmic scales, with log[$n(D)$] plotted against D.

For purposes of these simulations, (1) is more conveniently expressed in terms of the total drop number concentration N_T and the mass-weighted mean diameter D_m [= $(\mu+4)/\lambda$] as

$$n(D)=N_T\frac{(\mu+4)^{\mu+1}}{\mu!}\frac{D^{\mu}}{D_m^{\mu+1}}\exp\left[-(\mu+4)D/D_m\right] \quad (2)$$

Defining a dimensionless size variable $y = D/D_m$, this becomes

$$n(y)=N_T\frac{(\mu+4)^{\mu+1}}{\mu!}y^{\mu}\exp\left[-(\mu+4)y\right] \quad (3)$$

This can be recognized as the product of the mean number concentration N_T and the gamma probability density function (PDF) of drop size. For convenience herein, we also designate the mean sample size (number of drops) as N_T. This could be viewed as representing a volume sampling instrument with a sample volume of 1 m^3 (independent of the drop size). However, a sample volume of α m^3 and a mean drop concentration N_T/α would lead to the same mean sample size and the same sampling statistics. Thus, with the drop sizes normalized to D_m, the results can be organized simply by the values of N_T and the gamma shape parameter μ.

The simulations proceed from selected values of μ and N_T by first drawing from a Poisson distribution with mean value N_T to determine the actual total number of drops C in a given sample. Then C values of y drawn from the gamma PDF establish the (normalized) sizes of those drops. Normalized values for the six sample moments M_{1S} through M_{6S} are next calculated for each sample, and then various moment-based calculations (discussed in Sec. 4 and summarized in the Appendix) are applied to estimate the DSD parameters. For purposes of this work, we classified the drop sizes into intervals of $\Delta y = 0.02$, representing the size classification procedure common to drop-measuring instruments, and truncated the gamma PDF at $y = 3.0$. Repetition of the sampling and "fitting" process yields the desired distributions. We used about 1,000,000 drops (e.g. 50,000 samples with $N_T = 20$ and 5,000 samples with $N_T = 200$) in the simulations; as the probability of a drop in a gamma PDF with $\mu = 2$ being larger than $y = 3.0$ is 2.76×10^{-6} (and is even smaller for higher values of μ), we are lacking only a few larger drops from a full gamma DSD.

3. CHARACTERISTICS OF SAMPLING DISTRIBUTIONS

Before considering the moment-based "fits", we first examine some characteristics of the distributions of the sample moments

$$M_{iS}=\sum_{C}D^i$$

($i = 3$ gives the sample moment related to liquid water concentration, or LWC; $i = 6$ gives radar reflectivity factor Z) themselves. Sampling from long-tailed DSDs like the gamma exhibits certain general features. Sample values of the moments are unbiased: the expected, or mean, value of M_{iS} corresponds to that moment of the drop population being sampled. However, the sampling distributions are skewed, as indicated by the fractional standard deviations calculated by Gertzman and Atlas (1977) and as shown in Smith *et al.* (1993) or Smith and Kliche (2005). According to Gertzman and Atlas, the fractional standard deviation (FSD) σ_i/M_i for sample values of DSD moment M_i as determined with a volume-sampling instrument studying a gamma DSD would be

$$\frac{\sigma_i}{M_i}=\frac{1}{\sqrt{V_s}}\frac{\sqrt{M_{2i}}}{M_i} \quad (4)$$

where V_S is the instrument sampling volume. The general form for the moments of a gamma DSD can be written

$$M_i=N_T(\mu+1)(\mu+2)\ldots(\mu+i)\left[\frac{D_m}{\mu+4}\right]^i \quad (5)$$

Thus

$$\frac{\sigma_i}{M_i} = \frac{1}{\sqrt{N_T V_S}} \frac{\sqrt{(\mu+1)(\mu+2)\ldots(\mu+2i)}}{(\mu+1)(\mu+2)\ldots(\mu+i)} \qquad (6)$$

The product $N_T V_S$ in the denominator of (6) is the mean sample size, which for a sample volume of 1 m^3 would be numerically equal to N_T. Thus for i = 3 (LWC), μ = 2 and N_T = 50, (6) gives (σ_3/M_3) = 0.335. The distribution of a positive quantity (M_{3S}) with an FSD this large is necessarily skewed.

Inspection of (6) shows that the skewness increases with the order i of the moment M_i, and decreases with increasing sample size N_T or with increasing values of μ. Figure 1 illustrates the first property for the sample moments M_{3S} (LWC) and M_{6S} (Z), for the case μ = 2, N_T = 50. The general tendency is for the sample moments to be lower than the corresponding population values; this behavior is the ultimate cause of the bias in the moment methods for estimating DSD parameters. As shown in Sec. 4, the increase of the skewness with the moment order leads to greater biases when higher moments are employed. Figure 2 illustrates how the median sample moments approach the population values as the sample size increases. The skewness of the sampling distributions, and the resulting biases, decrease in a similar manner.

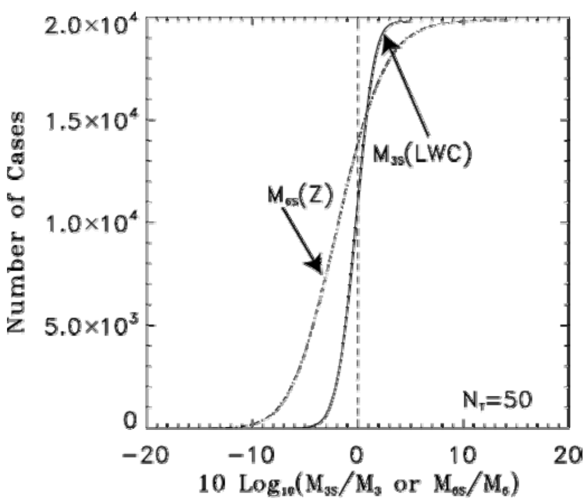

Fig. 1: Cumulative sampling distributions of third sample moment M_{3S} (proportional to LWC) and sixth moment M_{6S} (proportional to Z). Sample moments normalized with respect to the corresponding population value; vertical dashed line indicates population value. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

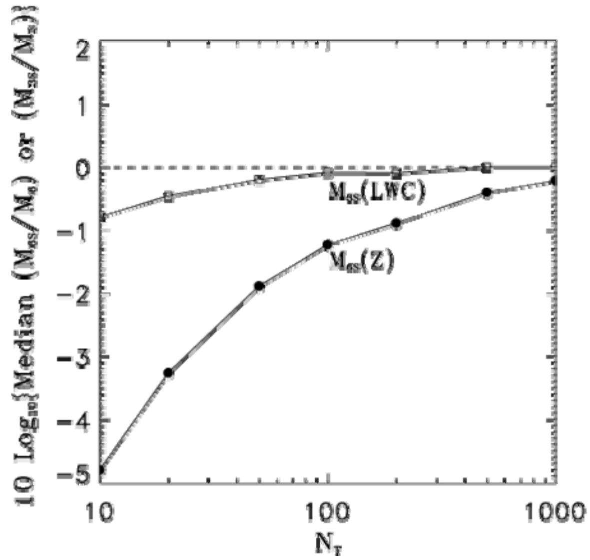

Fig. 2: Plot of median values of third sample moment M_{3S} (proportional to LWC) and sixth moment M_{6S} (proportional to Z) versus mean sample size; sample moments normalized with respect to the corresponding population value. Horizontal dashed line indicates population value. Population DSDs: gamma, μ = 2.

Sampling the small drops can be a major instrumental problem, and for gamma DSDs adequately sampling the relatively rare large drops is also an important concern. Regardless of the (population) value of μ, fewer than one drop in 129 in a gamma DSD is larger than D = 1.5 D_m, and fewer than one in 1900 is larger than D = 2 D_m. However, for μ = 2 the drops larger than 1.5 D_m contribute more than 10% of the LWC and almost half of the reflectivity factor. Consequently, the relatively large but relatively rare drops tend to be important in determining the moments of physical significance. The sample values of these moments are therefore correlated with the size of the largest drop in each sample (e.g., Fig. 3), though the correlations are weaker than corresponding ones found with an exponential population DSD (Smith and Kliche 2005). Table 1 demonstrates that these correlations are stronger for higher-order moments, and remain appreciable even for fairly large sample sizes. The sample moments are in turn correlated with each other, an artifact of the sampling variability as discussed in Smith *et al.* (1993) or Smith and Kliche (2005). Figure 4 illustrates this spurious correlation between moments M_4 (or R*, a surrogate for rainfall rate as suggested by Joss and Gori 1978) and M_6 (or radar reflectivity factor Z), for the case $\mu = 2$, $N_T = 50$.

The distribution of values along the abscissa in Fig. 3 here demonstrates that the maximum drop size in a gamma DSD is rarely approached (and this is true even in samples of hundreds of drops). There is clearly no basis for assuming truncation of the underlying DSD at the maximum observed drop diameter, with samples of such sizes.

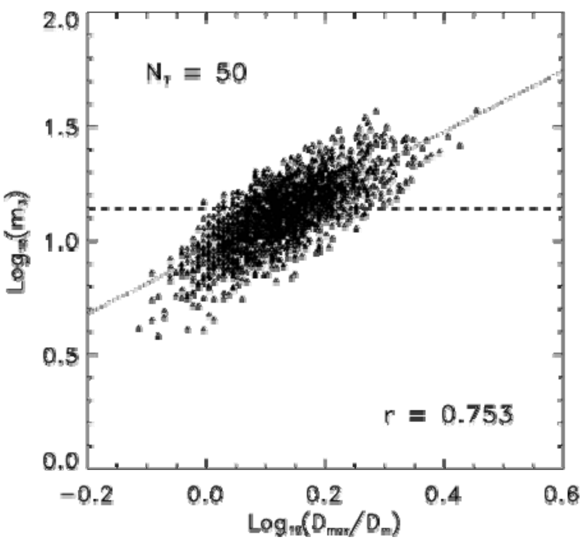

Fig. 3. Scatter plot of sample values of normalized third moment (m_3, proportional to LWC) against (normalized) maximum drop size in the sample. Horizontal dashed line indicates population value of m_3; dotted line shows regression relationship. Population DSD: gamma, $\mu = 2$. Mean sample size: 50 drops.

Table 1: Correlations between sample moments and maximum drop size in a sample.

Population DSD: gamma, $\mu = 2$

	Sample Moment			
Mean sample size (N_T)	M_{2S}	M_{3S}	M_{4S}	M_{6S}
10	0.772	0.906	0.950	0.982
20	0.682	0.846	0.917	0.969
50	0.559	0.760	0.866	0.949
100	0.453	0.674	0.820	0.927
200	0.381	0.596	0.753	0.902
500	0.311	0.506	0.674	0.865
1000	0.245	0.408	0.572	0.803

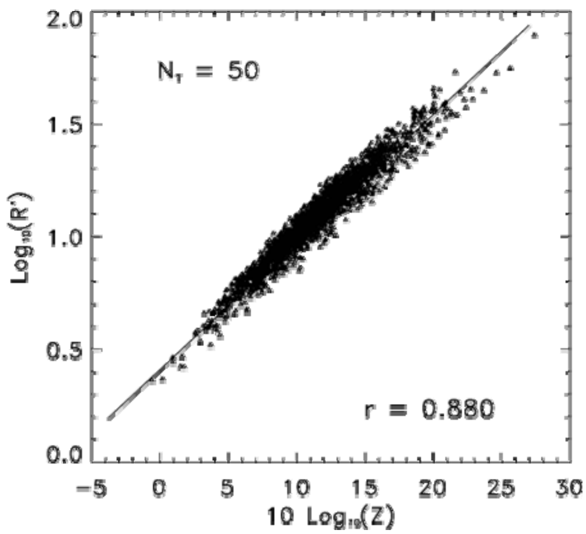

Fig. 4: Scatter plot of sample values of normalized fourth moment (m_4, proportional to R^*) against normalized sixth moment (m_6, proportional to Z). Solid line shows regression relationship; its slope corresponds to an exponent 1.75 in the customary Z-R relationship. Population DSD: gamma, $\mu = 2$. Mean sample size: 50 drops.

4. THE BIAS IN MOMENT ESTIMATORS

The essence of the moment approach for estimating parameters for DSD functions is to use the same number of moments calculated from observed raindrop size distributions as there are parameters in the function to be "fitted." Analytical expressions for the selected moments of that function are solved algebraically for the needed parameters, and observed values of the sample moments then entered into the resulting equations to estimate the parameters. The Appendix discusses the relevant mathematical expressions used here. The use of moment methods for rain DSDs evidently began with Waldvogel's (1974) paper on the "N_0 jump" of DSDs. He used observed values of moments M_{3S} and M_{6S} (i.e. LWC and Z) to determine pairs of parameters for exponential functions that purportedly represented the observed DSDs. However, most functions "fitted" in this way do not represent well either the samples upon which they are based or the underlying populations from which the samples were taken.

4a. Moment Estimators for Gamma Functions

Figure 5, based on an "ideal sample" of 50 drops from a gamma DSD with shape parameter μ = 2, illustrates the bias in the moment-method "fits." This "ideal sample," unlike the randomly-drawn samples used elsewhere in this paper, was constructed from the cumulative PDF of drop size using the systematic procedure described below. It provides as close a representation of the PDF as could be achieved with a sample of 50 drops. Construction of the sample began with the "inverse cumulative PDF" $N_L(D)$, indicating the number of drops of diameter D or larger; closed-form expressions do not exist for cumulants of the gamma distribution, but good numerical approximations are available for integer values of μ. To illustrate the procedure, we note that 48.17/50 of the drops in the population have $D/D_m \geq 0.12$, while (with the drop sizes quantized in intervals of $\Delta D/D_m = 0.02$) only 47.33/50 have $D/D_m \geq 0.14$. Thus we assigned $D/D_m = 0.13$ to one drop in the sample. Similarly, only 46.35/50 of the drops have $D/D_m \geq 0.16$, so $D/D_m = 0.15$ was assigned the next drop in the sample; and so on. The stairstep plot in Fig. 5 represents the resulting sample.

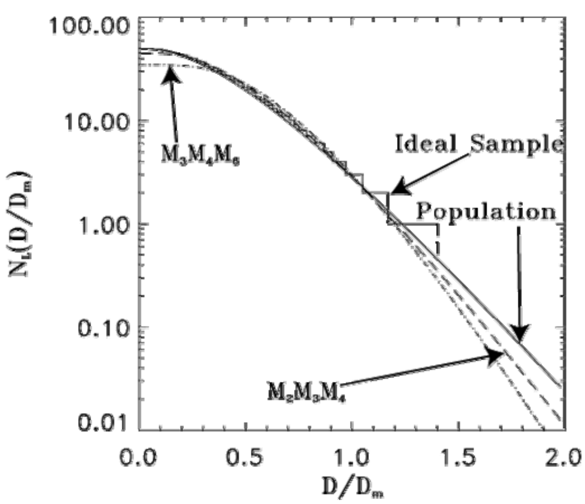

Fig. 5: Illustration of an "ideal sample" of 50 drops from a gamma DSD (μ = 2) along with two moment-based gamma "fits" to the sample; plot uses "inverse cumulative" $N_L(D)$ format advocated in Smith (1982). Solid line represents the underlying drop population, while stairstep plot represents the "ideal sample" constructed as described in text. Dashed and dash-dot lines represent "fits" based on moments M_2, M_3, M_4 and M_3, M_4, M_6 respectively.

The figure includes two moment-method "fits" to the sample for gamma DSD functions. One of the two is based on moments M_2, M_3 and M_4, as suggested in Smith (2003), and the other is based on M_3, M_4 and M_6, as used for example by Ulbrich (1983), Kozu and Nakamura (1991), or Tokay and Short (1996). The latter does not represent either the "observed" sample or the original population DSD very well; the smaller discrepancy resulting when the lower moments (M_2, M_3, and M_4) are used in the calculation is evident.

The foregoing discussion and the specific example in Fig. 5 suggest the general nature of the bias in moment estimators for parameters of gamma DSD functions: they tend to overestimate both the distribution shape parameter μ and the size (scale) parameter λ. In terms of the parameters of (2), they tend to underestimate both D_m and N_T – yielding "fits" having too few drops that are too small compared to the original raindrop population. Figure 6 illustrates the bias in moment estimates of D_m, for random samples from a population with μ = 2, N_T = 50. The biases are greater when higher moments are used in the procedure; Fig. 7 illustrates this behavior for $\hat{\mu}$. Consequently, procedures that use sample values of reflectivity in the moment calculations lead to greater biases than ones employing only lower moments.

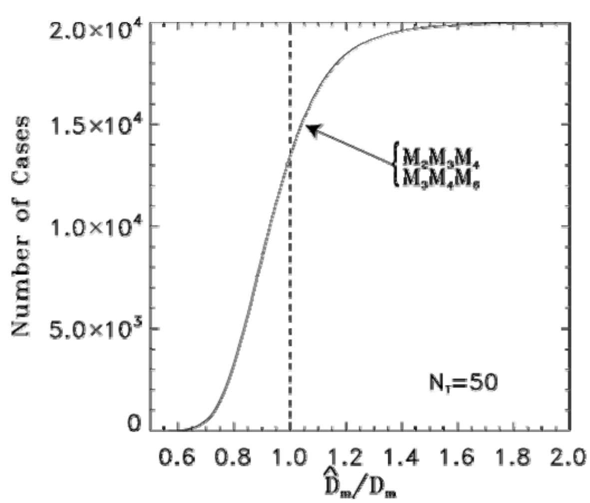

Fig. 6: Cumulative distribution of normalized values of population mass-weighted mean diameter, as estimated from each sample using the indicated sets of sample moments; vertical dashed line indicates population value. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

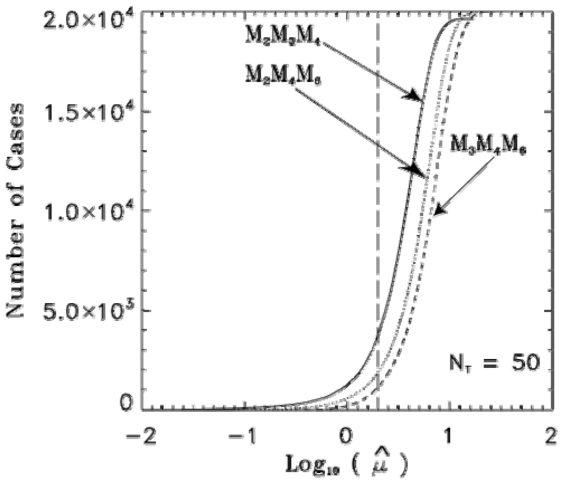

Fig. 7: Cumulative distributions of values of gamma DSD shape parameter $\hat{\mu}$ (log scale), as estimated from the indicated sets of three sample moments; vertical dashed line indicates population value. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

Extension of this argument would appear to suggest that using the lowest moments – M_{0S}, the sample size, M_{1S}, related to the mean drop diameter, and M_{2S} – would yield the smallest biases of all. In the case of exponential population DSDs discussed in Smith and Kliche (2005), inadequate instrument responses to very small drops mean that trying to use moments lower than M_{2S} in the analysis would introduce another kind of uncertainty into the moment procedures. However, in a gamma population with μ >1 or so there are fewer very small drops and their contribution to moments of order 1 or higher are less important. For example, in a gamma distribution with μ = 2 about 27% of the drops are smaller than D/D_m = 0.3; those small drops contribute about 11% of the value of M_1 but less than 4% of the value of M_2. Depending upon the precision required, use of M_{1S} with samples from gamma DSDs might be satisfactory.

Another formulation of the gamma DSD, involving a normalized concentration parameter N_W (Bringi and Chandrasekar 2001), has come into use:

$$n(D) = \frac{3}{128} N_W \frac{(\mu+4)^{\mu+4}}{(\mu+3)!} \left(\frac{D}{D_m}\right)^{\mu} \times \exp\left[-(\mu+4)D/D_m\right] \quad (7)$$

The moment estimators tend to overestimate the value of N_w, as illustrated in Fig. 8.

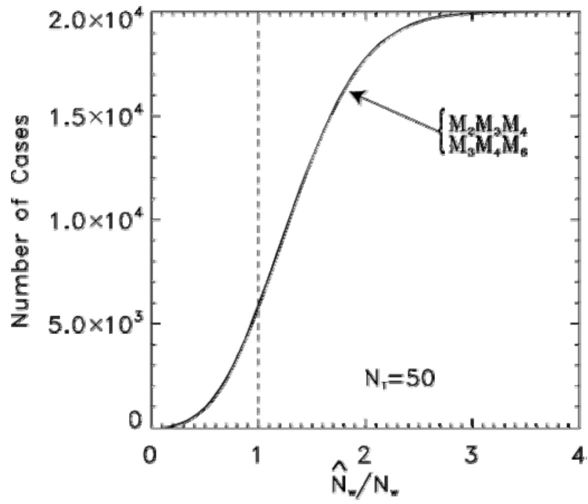

Fig. 8: Cumulative distribution of values of normalized gamma DSD concentration parameter $\hat{N}_w$, as estimated from the indicated sets of three sample moments. Abscissa indicates ratio of sample estimate to population value; vertical dashed line denotes population value. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

The skewness in the sampling distribution for the moments diminishes as the sample size increases (e.g. Fig. 2), so the bias in the moment estimators should also decrease with increasing sample size. Figure 9 shows that to be the case; with samples of hundreds or thousands of drops the bias may become small enough to be negligible for many purposes.

4b. Comparison with Maximum Likelihood Estimators

The maximum likelihood (ML) estimators advocated by Haddad *et al.* (1996, 1997) should provide more accurate estimates of the gamma parameters, even though the ML estimators have some bias (Choi and Wette 1969). The ML estimates for μ are obtained by solving

$$\ell n(\hat{\mu}_{MLE}+1) - \psi(\hat{\mu}_{MLE}+1) = \ell n\left[\frac{\bar{y}}{\left(\prod_{i=1}^{n} y_i\right)^{1/n}}\right] \quad (8)$$

where ψ is the "psi" or "digamma" function defined by $\psi = \Gamma'(x)/\Gamma(x)$. This formulation does require good estimates of $\bar{y}$, the ratio of M_{1S}/M_{0S}, which can be difficult to obtain in practice. An iterative solution to (8) based on Bowman and Shenton (1988) has been used to compute some ML estimates of μ for comparison with the moment estimates. Figure 10 shows that the variance of the ML estimates tends to be much smaller than that of the moment estimates in this idealized simulation. Moreover, the mean ML estimate of $\hat{\mu}$ = 2.17 is considerably closer to the population value (μ = 2) than are the mean moment estimates ($\hat{\mu}$ = 3.79 for M_2, M_3, M_4 and 3.34 for M_0, M_1, M_2). The degree to which practical constraints in measuring $\bar{y}$ accurately would degrade the ML estimates remains to be established.

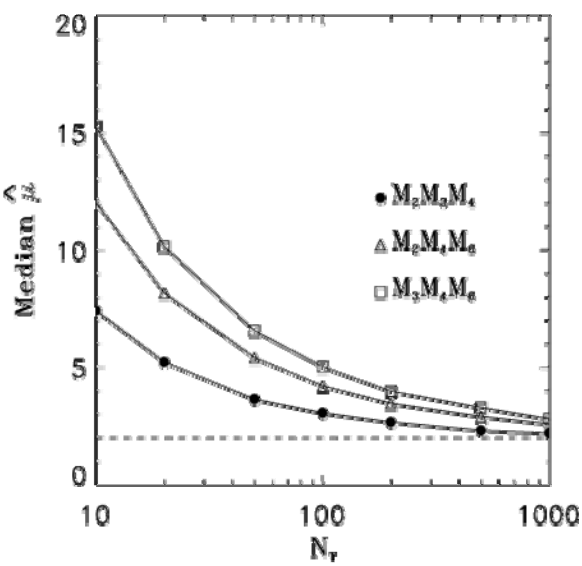

Fig. 9: Variation of median value of gamma DSD shape parameter $\hat{\mu}$, as estimated from the indicated sets of three sample moments, with mean sample size. Population DSD: gamma, μ = 2. Horizontal dashed line indicates population value.

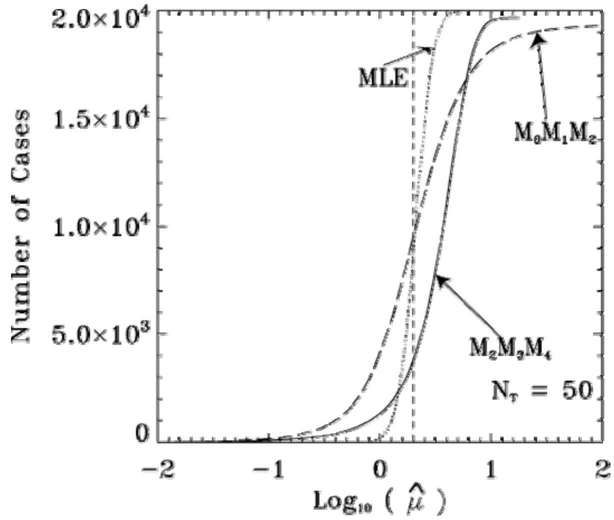

Fig. 10: Cumulative distributions of values of gamma DSD shape parameter $\hat{\mu}$ (log scale), as estimated by maximum likelihood methods (MLE curve) or the indicated sets of three sample moments; vertical dashed line indicates population value. The M_0, M_1, M_2 values are included for comparison with the ML estimates, since the latter make use of sample values of M_{0S} (C) and M_{1S}. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

5. RELATED FINDINGS

Though the various sample moments are correlated with the maximum drop size in a sample (Sec. 3), and there are associated correlations between moments (e.g. Fig. 4), the correlation of the "fitted" parameters with the maximum drop size found when the population DSD is exponential in form (Smith and Kliche 2005) does not appear in the present simulations. For example, with an exponential population DSD the correlation between estimated size (slope) parameter $\hat{\Lambda}$ and the sample D_{max} was -0.929 when N_T = 50; with a gamma (μ = 2) population DSD the corresponding correlation was only -0.007. Nevertheless, the "fitted" parameters are still correlated with each other; Zhang *et al.* (2003) noted this type of correlation between parameters $\hat{\mu}$ and $\hat{\lambda}$ and it also appears in these simulations. Figure 11 shows a similar correlation between $\hat{N}_w$ and $\hat{D}_m$. Such correlations (caused here entirely by sampling variability) could be mistaken for physical relationships. The correlations between estimated parameters are not much weaker when lower-order moments are used in the "fitting" process; with N_T = 50, the log $\hat{N}_w - \hat{D}_m$ correlation is -0.901 when sample moments M_{2S}, M_{3S}, and M_{4S} are used compared to the -0.904 value with moments M_{3S}, M_{4S}, and M_{6S} as illustrated in Fig. 11.

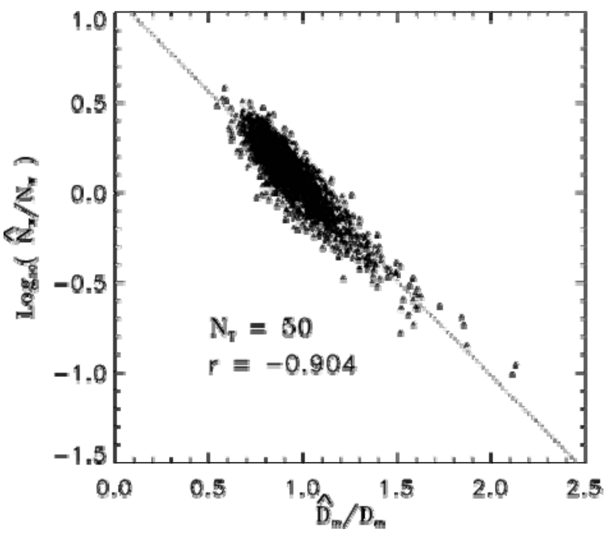

Fig. 11: Scatter plot of values of normalized gamma DSD concentration parameter N_W (log scale) versus size parameter D_m, using estimates based on sample moments M_3, M_4, and M_6; sample values normalized to corresponding population values. Dotted line indicates regression relationship. Population DSD: gamma, μ = 2. Mean sample size: 50 drops.

The log $\hat{N}_w - \hat{D}_m$ relationship becomes more nearly linear as the sample size increases, and the correlation coefficient remains stronger than -0.90 (for moments M_{3S}, M_{4S}, M_{6S}) when $N_T \geq 50$ – though the ranges of variation of the parameters decrease. In any case, this behavior suggests that one should be wary of inferring physical relationships between such "fitted" parameters until the effects of the sampling variability have been taken into account.

6. IMPLICATIONS FOR ANALYSIS OF EXPERIMENTAL DATA

In trying to relate these simulations to actual raindrop observations, one should keep in mind several factors:

- The actual population DSDs in nature are unknown. There is no assurance that they are gamma, and even if they are the shape parameter is unknown and may vary.
- Observations with surface-sampling instruments, such as impact disdrometers, involve sample volumes that increase with the drop size – which tends to mitigate the skewness in the high-order sample moments, and consequently the associated biases.
- The observations include only the actual sample size (*C* in these simulations). The mean, or expected, sample size (N_T) is not known, though the actual sample size provides a better approximation as it increases.
- Very small drops are generally absent from many such observations – either because such small drops are not present, or because the instruments do not respond to those drops. In a gamma DSD with μ = 2, 12% of the drops would be smaller than 0.2 D_m and 58% would be smaller than 0.5 D_m. With typical values of D_m being 1-3 mm, this means the simulations may involve more than perhaps twice the total numbers of drops that would be found in corresponding observations. Thus results given above for N_T = 50 might be more applicable to observations with total drop counts of, say, 25.

These caveats notwithstanding, certain broad inferences are applicable:

- Values of the parameters for DSD functions as estimated by moment methods will be biased.
- The bias will be stronger when higher moments are employed.
- The bias will diminish as the sample size increases.

Thus DSD parameters estimated using high-order sample moments (e.g., reflectivity) with small sample sizes (say a few tens of drops) are the most highly suspect. Moreover, as suggested in the preceding section, caution should be used in attempting to impute a physical basis to relationships between DSD parameters “fitted” by moment methods.

As the actual DSD parameters (even if the DSD were gamma) are unknown, it is difficult to critique in a quantitative manner any given set of results based on moment-method analyses. The most likely indication of the kind of biases discussed here appears in published frequency distributions of the gamma shape parameter $\hat{\mu}$, such as those reported in Kozu and Nakamura (1991) or Tokay and Short (1996). They extend to values as large as 30, which in view of Fig. 7 herein may well be a consequence of the M_3, M_4, M_6 moment approach used by those authors. The fact that the maximum $\hat{\mu}$ values reported by Kozu and Nakamura decrease as the rainfall rate increases (which should be accompanied by increases in sample size) lends further weight to this interpretation. A simple way to test this idea would be to stratify the shape-parameter estimates by sample size, to look for a trend similar to that shown in Fig. 9.

7. CONCLUSIONS

Moment estimators for parameters of DSD functions are inherently biased. They tend to give erroneous values of the DSD parameters unless the drop samples are much larger than those commonly available. In particular, estimates of the gamma shape parameter μ tend to be far larger than the shape parameter of the underlying DSD from which the samples are taken. The bias is strongest for small sample sizes, and also stronger when higher-order moments of the observed DSDs are used in the “fitting” process.

Moment methods may provide estimates of DSD parameters of sufficient accuracy if very large samples (hundreds, perhaps thousands) of drops are available. Failing that, some alternative approach to fitting the observed DSDs must be used. The maximum likelihood approach suggested by Haddad *et al.* (1996, 1997) may be satisfactory, though the maximum likelihood estimators are not without bias (Choi and Wette 1969). They may also run afoul of practical problems with the DSD observations.

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Appendix: Equations for Moment Estimators of Gamma DSD Parameters

This appendix shows how the expressions used to calculate the normalized estimates of gamma DSD parameters summarized in this paper are developed. To illustrate the moment-method procedures as employed in these simulations, consider the case of parameters N_T, μ and D_m for a gamma DSD function to be estimated from sample moments M_{2S}, M_{3S} and M_{4S}. From (5),

$$M_2 = N_T(\mu+1)(\mu+2)D_m^2/(\mu+4)^2$$

$$M_3 = N_T(\mu+1)(\mu+2)(\mu+3)D_m^3/(\mu+4)^3$$

$$M_4 = N_T(\mu+1)(\mu+2)(\mu+3)D_m^4/(\mu+4)^3$$

Algebraic solution yields the relationships

$$N_T = \frac{M_2^2}{M_4}\ \frac{\alpha}{(2-3\alpha)(1-2\alpha)}$$

$$\mu = \frac{(3-4\alpha)}{(\alpha-1)}$$

$$D_m = M_4/M_3$$

with

$$\alpha = M_3^2/(M_2 M_4)$$

The sample moments in these simulations are

$$M_{iS} = m_i D_m^i \tag{A-8}$$

where

$$m_i = \sum_C y^i \tag{A-9}$$

Substituting the sample moments into (A-4)

through (A-7) yields the relationships

$$\hat{N}_T = \frac{m_2^2}{m_4} \frac{\hat{\alpha}}{(2-3\hat{\alpha})(1-2\hat{\alpha})} \tag{A-10}$$

$$\hat{\mu} = \frac{(3-4\hat{\alpha})}{(\hat{\alpha}-1)} \tag{A-11}$$

$$\hat{D}_m = m_4 D_m / m_3 \tag{A-12}$$

with $$\hat{\alpha} = m_3^2 / (m_2 m_4) \tag{A-13}$$

From (A-12),

$$\hat{D}_m / D_m = m_4 / m_3 \tag{A-14}$$

In this fashion, parameter estimates based on the normalized sample moments m_i can be compared to the population parameters in dimensionless expressions where the actual drop sizes do not enter. (The gamma concentration parameter n_1 is an exception; moment estimates of that quantity cannot be conveniently normalized.) Thus the only population parameters that enter the simulations are the DSD shape parameter μ and the mean sample size N_T. Expressions based on other combinations of DSD parameters and sample moments appear in the Appendix of Smith and Kliche (2005).

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