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ON A THEORY OF RELATIVE DISPERSION BY CONTINUOUS MOVEMENTS

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1. INTRODUCTION

Relative dispersion indicates the process of distribution of marked fluid particles around their instantaneous centre of mass. For instance, the expansion of a puff of particles is a typical phenomenon of relative dispersion.

A fundamental quantity in relative dispersion studies is the mean square separation σ_r^2 between pairs of particles, which is a measure of the size of the puff. Based on dimensional arguments, Obukhov (1941) proposed the relationship $\sigma_r^2 = C_r \varepsilon\, t^3$ for the inertial subrange of turbulence, where ε is the mean dissipation of kinetic energy, and C_r the Richardson-Obukhov constant, whose value is still uncertain.

The original theory of Taylor (1921) refers to *absolute* dispersion, i.e. it provides the standard deviation σ_y of a distribution of particles with respect to a *fixed* origin, in statistically homogeneous turbulent flow. Typically, σ_y describes the length of the cross-wind arc spanned by a plume for a virtually infinite sampling time as measured by fixed samplers at a certain distance from the source, assuming constant mean wind direction. Note that σ_y does not provide any information on the actual, instantaneous, width of the plume, which is related to σ_r.

The application of Taylor's theory to derive relative dispersion standard deviations presents several difficulties. Perhaps the most serious problem is that the autocovariance of relative velocity is a function of time, as well as of the time lag. Thus its parameterizations, as well as its experimental observations, are nontrivial tasks.

We derive a differential equation for σ_r^2, which can be solved numerically, and has an analytical solution in the inertial subrange limit, consistent with Obukhov's solution. The derivation is based on the application of Taylor's (1921) statistical diffusion theory to relative dispersion, and on the definition of the turbulent kinetic energy of separation.

Our theory predicts an analytical expression of C_r. It also provides the definition of decorrelation time scale for relative dispersion, which is shown to be the fundamental quantity in the calculation of σ_r^2, as opposed to the autocorrelation function. The results are extended to finite Reynolds number turbulence using relationships based on a Reynolds number-dependent Lagrangian time scale. Several predictions are compared to direct numerical simulations (DNS) and laboratory observations.

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2. THEORY

Position and velocity of a particle relative to the centre of mass of a cluster are defined as $\boldsymbol{y}_r = \boldsymbol{y} - \overline{\boldsymbol{y}}$ and $\boldsymbol{v}_r = \boldsymbol{v} - \overline{\boldsymbol{v}}$ respectively, where the overbar represents average over all particles of the cluster. Averages over the ensemble of realizations are represented by angled brackets.

The component of the mean square distance of the particles from their respective instantaneous centre of mass over an ensemble of realizations, along an arbitrary y-axis, will be indicated as $\sigma_{yr}^2 \equiv \langle y_r^2 \rangle$, and the mean square relative velocity as $\sigma_{vr}^2 \equiv \langle v_r^2 \rangle$.

σ_{yr}^2 satisfies the differential equation:

$$\frac{d\sigma_{yr}^2}{dt} = 2\left\langle y_r \frac{dy_r}{dt} \right\rangle = 2\sigma_{vr}^2(t) \int_0^t R_r(t,\tau)d\tau \qquad (1)$$

where the release time $t_o = 0$ for simplicity, and the autocorrelation coefficient R_r of Lagrangian relative velocity v_r was defined as

$$R_r(t,\tau) = \frac{\langle v_r(t) v_r(t-\tau) \rangle}{\sigma_{vr}^2(t)}, \qquad (2)$$

with $0 \leqslant \tau \leqslant t$.

In the inertial subrange we can write (Franzese, 2003):

$$\sigma_{vr}^2 = \sigma_v^2 \left(\frac{\sigma_{yr}}{\sigma_{yL}} \right)^{2/3} \qquad (3)$$

where σ_v^2 is the velocity variance, and σ_{yL} is a length scale for the energy-containing eddies. Because the Lagrangian *relative* velocity structure function $\left\langle [v_r(t) - v_r(t-\tau)]^2 \right\rangle = C_o \varepsilon \tau$, where the constant C_o is the same as in the Lagrangian *absolute* velocity structure function, Eq. (2) becomes:

$$R_r(t,\tau) = 1 - \frac{C_o \varepsilon}{2\sigma_{vr}^2(t)}\tau - \frac{1}{2t}\tau \qquad (4)$$

which defines the relative dispersion time scale

$$T_r^{-1} = \frac{C_o \varepsilon}{2\sigma_{vr}^2(t)} + \frac{1}{2t} \qquad (5)$$

2.1 *Analytical solution*

Substituting (4) in (1), we write the solution

$$\sigma_{yr}^2 = 2\int_0^t \sigma_{vr}^2(t) T_r(t) dt = C_{yr} \varepsilon t^3 \qquad \text{for } t \leqslant T_{r_L} \qquad (6)$$

$$\sigma_{yr}^2 = \sigma_{yL}^2 + 2\sigma_v^2 T_L (t - T_{r_L}) \qquad \text{for } t \geqslant T_{r_L} \qquad (7)$$

with $C_{yr} = C_r/6$, where C_r is the well-known Obukhov-Richardson two-particle relative dispersion constant, and

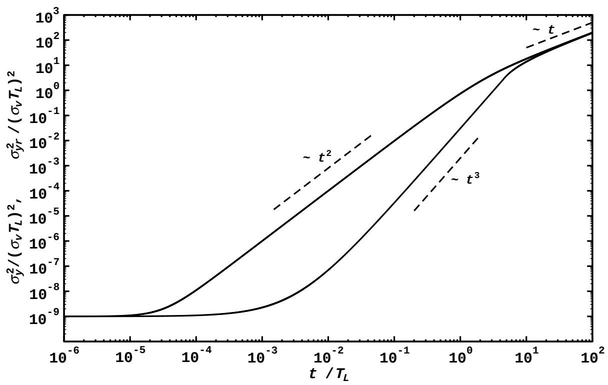

FIG. 1: The evolutions of the absolute (i.e. Taylor's σ_y) and relative [i.e. σ_{yr} from Eqs. (6) and (7)] dispersion variances for an initial source $\sigma_{yo}^2 = 10^{-9}(\sigma_v T_L)^2$. The straight lines are proportional to t, t^2 and t^3.

$T_{r_L} = \beta T_L$, where T_L is the familiar Lagrangian time scale, and $\beta \equiv \beta(\sigma_v, \sigma_{yL}, T_L)$.

After some algebra it is possible to obtain an explicit relationship between C_r and C_o:

$$C_r = \alpha C_o, \text{ with } \alpha \equiv \alpha(\sigma_v, \sigma_{yL}, T_L) \tag{8}$$

The theory predicts $\alpha \simeq 1/11$ if the large-eddy scale σ_{yL} is taken equal to σ_y at the onset of the large-time Brownian diffusion regime (i.e. when $\sigma_y \propto \sqrt{t}$) for a linear autocorrelation function. Thus a typical value of $C_o = 7$ gives $C_r = 0.64$. Figure 1 shows the absolute (i.e. Taylor's equation for σ_y) and relative [i.e. Eqs. (6) and (7)] dispersion variances normalized over $(\sigma_v T_L)^2$ for an initial source $\sigma_{yo}^2 = 10^{-9}(\sigma_v T_L)^2$. The figure shows the smooth transition of σ_{yr}^2 between different scaling regimes, and the consistency between σ_y^2 and σ_{yr}^2 at the asymptotic limits. The correct large time behaviour of σ_{yr}^2 is a natural result of the theory - not a consequence of *ad hoc* a priori assumptions.

T_r in the inertial subrange can be expressed as:

$$T_r = 2t/(1 + 4\alpha^{-1/3}) \simeq 0.2t \tag{9}$$

The linear dependence of T_r on t implies that $R_r(t, \tau)$ is a function of the single variable τ/t. The predicted $R_r(t, \tau) = \exp(-\tau/T_r)$ as a function of τ/t is plotted in figure 2 along with the results of the experiments in two dimensional turbulence reported in Jullien et al. (1999). It can be seen that the predicted dependence of $R_r(t, \tau)$ on τ/t is well supported by the observations.

3. REYNOLDS NUMBER EFFECTS

Sawford (1991) found that the effective value of C_o as a function of the Taylor-scale Reynolds number Re_λ is well approximated by:

$$\widetilde{C}_o = C_o/(1 + 7.5C_o^2 Re_\lambda^{-1.64}) \tag{10}$$

where $\widetilde{C}_o$ is the value of C_o at finite Re_λ, and C_o was estimated to be about 7 based on comparisons with DNS

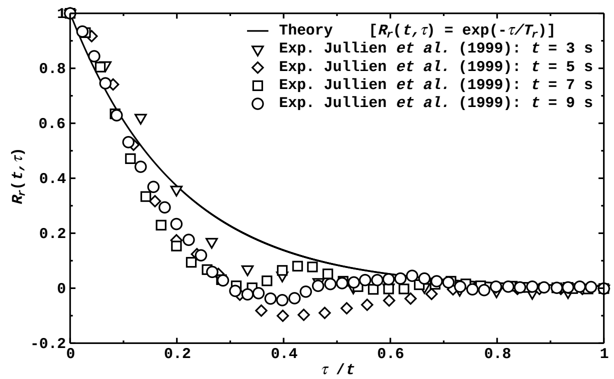

FIG. 2: Relative velocity autocorrelation function $R_r(t, \tau)$

data. As a consequence of Eq. (8), the dependence of C_r on Re_λ is simply written as:

$$\widetilde{C}_r = \alpha C_o/(1 + 7.5C_o^2 Re_\lambda^{-1.64}) \tag{11}$$

The data from the experiments in grid turbulence by Ott and Mann (2000), and from the DNS of Ishihara and Kaneda (2002), Boffetta and Sokolov (2002) and Biferale et al. (2005) are reported in figure 3 along with Eq. (11). Note that experimental and numerical estimates of C_r are at present still uncertain, and a definitive conclusion on the best estimate of C_r cannot be drawn from the available data.

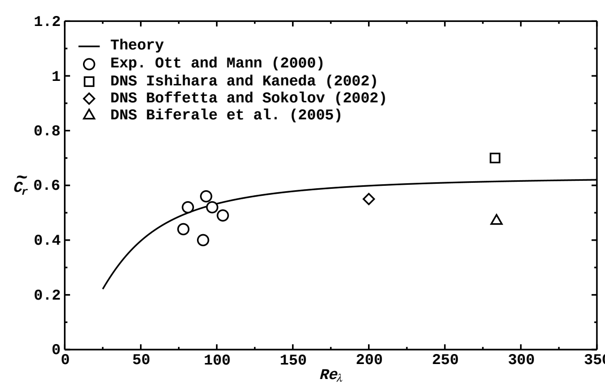

FIG. 3: Observed, simulated and predicted values of C_r as a function of Re_λ

4. CONCLUSIONS

Equations for mean square relative separation have been derived from first principles. The equations are consistent with Taylor's (1921) absolute dispersion theory, and with Obukhov's (1941) dimensional analysis.

The simplicity of the approach allows for a detailed analysis of dispersion dynamics, including the complementary process of plume meandering. Several quantities specific to relative dispersion, such as the relative velocity autocorrelation function R_r, the relative dispersion time scale T_r, and the relative separation energy σ_{vr}^2 are identified. These variables have been recently applied to define the dissipation of concentration fluctuations in PDF micromixing models of dispersion (Cassiani et al., 2005).

The results are extended to different Reynolds numbers using relationships based on Sawford's (1991) definition of a *Re*-dependent Lagrangian time scale.

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