1. INTRODUCTION

Relative dispersion indicates the process of distribution of marked fluid particles around their instantaneous centre of mass. For instance, the expansion of a puff of particles is a typical phenomenon of relative dispersion.

A fundamental quantity in relative dispersion studies is the mean square separation $\sigma^2_r$ between pairs of particles, which is a measure of the size of the puff. Based on dimensional arguments, Obukhov (1941) proposed the relationship $\sigma^2_r = C_r \varepsilon^3$ for the inertial subrange of turbulence, where $\varepsilon$ is the mean dissipation of kinetic energy, and $C_r$ the Richardson-Obukhov constant, whose value is still uncertain.

The original theory of Taylor (1921) refers to absolute dispersion, i.e. it provides the standard deviation $\sigma_y$ of a distribution of particles with respect to a fixed origin, in statistically homogeneous turbulent flow. Typically, $\sigma_y$ describes the length of the cross-wind arc spanned by a plume for a virtually infinite sampling time as measured by fixed samplers at a certain distance from the source, assuming constant mean wind direction. Note that $\sigma_y$ does not provide any information on the actual, instantaneous, width of the plume, which is related to $\sigma_r$.

The application of Taylor’s theory to derive relative dispersion standard deviations presents several difficulties. Perhaps the most serious problem is that the autocovariance of relative velocity is a function of time, as well as of the time lag. Thus its parameterizations, as well as its experimental observations, are nontrivial tasks.

We derive a differential equation for $\sigma^2_r$, which can be solved numerically, and has an analytical solution in the inertial subrange limit, consistent with Obukhov’s solution. The derivation is based on the application of Taylor’s (1921) statistical diffusion theory to relative dispersion, and on the definition of the turbulent kinetic energy of separation.

Our theory predicts an analytical expression of $C_r$. It also provides the definition of decorrelation time scale for relative dispersion, which is shown to be the fundamental quantity in the calculation of $\sigma^2_r$, as opposed to the autocorrelation function. The results are extended to finite Reynolds number turbulence using relationships based on a Reynolds number-dependent Lagrangian time scale. Several predictions are compared to direct numerical simulations (DNS) and laboratory observations.

2. THEORY

Position and velocity of a particle relative to the centre of mass of a cluster are defined as $y = y - \bar{y}$ and $v = v - \bar{v}$ respectively, where the overbar represents average over all particles of the cluster. Averages over the ensemble of realizations are represented by angled brackets. The component of the mean square distance of the particles from their respective instantaneous centre of mass over an ensemble of realizations, along an arbitrary $y$-axis, will be indicated as $\langle \sigma^2_y \rangle$, and the mean square relative velocity as $\langle \sigma^2_v \rangle$.

$\sigma^2_y$ satisfies the differential equation:

$$\frac{d\sigma^2_y}{dt} = 2\langle y^2 v^2 \rangle + 2\sigma^2_y(t) \int_0^t R_r(t, \tau) d\tau$$

where the release time $t_0 = 0$ for simplicity, and the autocorrelation coefficient $R_r$ of Lagrangian relative velocity $v_r$ was defined as

$$R_r(t, \tau) = \frac{\langle v_r(t) v_r(t - \tau) \rangle}{\langle v_r^2(t) \rangle},$$

with $0 \leq \tau \leq t$.

In the inertial subrange we can write (Franzese, 2003):

$$\sigma^2_r = \sigma^2_v \left( \frac{\sigma^2_r}{\sigma^2_y} \right)^{2/3}$$

where $\sigma^2_v$ is the velocity variance, and $\sigma^2_y$ is a length scale for the energy-containing eddies. Because the Lagrangian relative velocity structure function $\langle [v_r(t) - v_r(t - \tau)]^2 \rangle = C_0 \varepsilon^2 \tau$, where the constant $C_0$ is the same as in the Lagrangian absolute velocity structure function, Eq. (2) becomes:

$$R_r(t, \tau) = 1 - \frac{C_0 \varepsilon}{2\sigma^2_y(t)} \tau - \frac{1}{2l} \tau$$

which defines the relative dispersion time scale

$$T_r^{-1} = \frac{C_0 \varepsilon}{2\sigma^2_y(t)} + \frac{1}{2l}$$

2.1 Analytical solution

Substituting (4) in (1), we write the solution

$$\sigma^2_r = 2\int_0^t \sigma^2_v(t) T_r(t) dt = C_r \varepsilon \tau^2 \quad \text{for } t \leq T_L$$

$$\sigma^2_r = \sigma^2_{2L} + 2\sigma^2_v(T_L(t - T_L)) \quad \text{for } t \geq T_L$$

with $C_r = C_r/6$, where $C_r$ is the well-known Obukhov-Richardson two-particle relative dispersion constant, and
The data from the experiments in grid turbulence by Ott and Mann (2000), and from the DNS of Ishihara and Kaneda (2002), Boffetta and Sokolov (2002) and Biferale et al. (2005) are reported in figure 3 along with Eq. (11). Note that experimental and numerical estimates of $C_r$ are at present still uncertain, and a definitive conclusion on the best estimate of $C_r$ cannot be drawn from the available data.

4. CONCLUSIONS

Equations for mean square relative separation have been derived from first principles. The equations are consistent with Taylor’s (1921) absolute dispersion theory, and with Obukhov’s (1941) dimensional analysis.

The simplicity of the approach allows for a detailed analysis of dispersion dynamics, including the complementary process of plume meandering. Several quantities specific to relative dispersion, such as the relative velocity autocorrelation function $R_r$, the relative dispersion time scale $T_r$, and the relative separation energy $\sigma_{yr}^2$ are
identified. These variables have been recently applied to define the dissipation of concentration fluctuations in PDF micromixing models of dispersion (Cassiani et al., 2005).

The results are extended to different Reynolds numbers using relationships based on Sawford’s (1991) definition of a Re-dependent Lagrangian time scale.

REFERENCES


