Kelvin Waves in the Nonlinear Shallow Water Equations on the Sphere: Nonlinear Traveling Waves and the Corner Wave Bifurcation

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Two Topics

1. New asymptotic approximation for LINEAR Kelvin waves on the sphere
2. Corner wave bifurcation for NONLINEAR Kelvin waves on the sphere

Restrictions

1. Nonlinear shallow water equations
2. No mean currents

Two Parameters

1. Integer zonal wavenumber \( s \)
   \[ s > 0 \]
2. Lamb’s Parameter \( \epsilon \equiv \frac{4\Omega^2a^2}{gH} \)

\[ \Omega = \frac{2\pi}{84,600} \text{ s} \]
\[ a = \text{earth’s radius} \]
\[ g = 9.8 \text{ m/s} \]
\[ H = \text{equivalent depth} \]
Parametric range is SEMI-INFINITE in BOTH $s$ & $\epsilon$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Description</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.012</td>
<td>External mode: Venus</td>
<td>Lindzen (1970)</td>
</tr>
<tr>
<td>6.5</td>
<td>External mode: Mars</td>
<td>Zurek (1976)</td>
</tr>
<tr>
<td>12.0</td>
<td>External mode: Earth (7.5 km equivalent depth)</td>
<td>Lindzen (1970)</td>
</tr>
<tr>
<td>2.6</td>
<td>Jupiter: simulate Galileo data</td>
<td>Williams (1996)</td>
</tr>
<tr>
<td>21.5</td>
<td>Jupiter</td>
<td>Williams (1996)</td>
</tr>
<tr>
<td>43.0</td>
<td>Jupiter</td>
<td>Williams (1996)</td>
</tr>
<tr>
<td>260</td>
<td>Jupiter</td>
<td>Williams (1996)</td>
</tr>
<tr>
<td>87,000</td>
<td>ocean: first baroclinic mode (1 m equiv. depth)</td>
<td>Moore &amp; Philander (1977)</td>
</tr>
<tr>
<td>&gt; 100,000</td>
<td>ocean: higher baroclinic modes</td>
<td>Moore &amp; Philander (1977)</td>
</tr>
</tbody>
</table>

LINEAR EIGENFUNCTIONS are “HOUGH” functions (S. S. Hough (1896, 1898)

“We regard Mr. Hough’s work as the most important contribution to the dynamical theory of the the tides since the time of Laplace.”


Figure 1: Sydney Samuel Hough, FRS, (1870-1923)
Kelvin Parameter Space & LINEAR
Asymptotic Regimes

Small $\epsilon$: $u = P_s^{s}(\cos(\theta)) \exp(is\lambda - i\sigma t)$
(Longuet-Higgins, 1968))
Large $\epsilon$: $u \sim \exp(-\sqrt{\epsilon}\theta^2) \exp(is\lambda - i\sigma t)$
New asymptotic approximation is \((\mu = \sin(\text{latitude}))\)

\[
\phi \approx (1 - \mu^2)s^{\frac{1}{2}} \exp\left(-\left(\frac{1}{2}\right)\left\{\sqrt{\epsilon + s^2} - s\right\}\mu^2\right)
\]

Approximation (thick solid curve) and exact (thin solid curve) are GRAPHICALLY INDISTINGUISHABLE for \(s = 5, \epsilon = 5\)
Uniform Validity

- New approximation is uniformly valid for
  \[ \sqrt{s^2 + \epsilon} >> 1 \]
  (shaded in figure)

- Though not strictly valid when both \( s \) and \( \epsilon \) are \( O(1) \), it is not a bad approximation.
Barotropic ($\epsilon = 0$) Kelvin Waves

Equatorial trapping is not just due to $\epsilon$.
High zonal wavenumber Kelvin are equatorial modes even for $\epsilon = 0$

Barotropic Kelvin, $\epsilon=0$, $s=20$
Barotropic ($\epsilon = 0$) Kelvin Waves

Half-width is inside the tropics for $s \geq 5$

Figure 2:
Weakly Nonlinear Wave Theory, A Science Fiction Story

• Perturbative theory yields approximations that have the structure of LINEAR Kelvin waves in LATITUDE & DEPTH multiplied by a function $A(x, t)$ that solves a nonlinear PDE

$$u = h = A(x, t) \exp(-0.5y^2)\phi(z); \quad v \equiv 0$$

• On equatorial beta-plane without currents, Kelvin is dispersionless:

$$A_t + A_x + 1.22 AA_x = 0 \quad [1D \text{ Advection Eq}]$$ (1)

• Breaking and frontogenesis; no steady propagation

velocity scale is about 2 m/s

length scale is about 300 km
Kelvin Fronts & Breaking

Front curvature due to Kelvin-gravity wave resonant

Microbreaking

\[ A_t + \frac{1}{100} A_{xx} = A_x, \quad A(x,0) = -\sin(x) \]
Shear Currents: Dispersion

• Equatorial ocean has strong currents: South Equatorial Current (westward), North Equatorial Countercurrent (eastward), etc.

• Currents induce **dispersion** in Kelvin wave

• (Boyd, *Dyn. Atmos. Oceans*, 1984)

\[
A_t + 1.22 AA_x + dA_{xxx} = 0 \text{[KdV Eq]}
\]

KdV Predictions:

1. Solitons & cnoidal waves of **ARBITRARILY LARGE** amplitude

2. No breaking or frontogenesis
Difficulties with KdV Picture for Kelvin Wave

- KdV dispersion relation predicts a PARABOLA for the group velocity: true for $k < 1$ only
- As $k \to \infty$, $c_g \to c_{phase} \to$ constant
- Kelvin is WEAKLY DISPERSIVE for large zonal wavenumber (Boyd, *JPO*, 2005)
- KdV Theory FAILS at LARGE AMPLITUDE because shear-induced dispersion is TOO WEAK to balance NONLINEAR STEEPENING
Solitons on the Union Canal

Solitary waves were discovered observationally by John Scott Russell in the 1830’s. Below is a modern recreation on the same canal.

Figure 3:
What Really Happens: CCB Scenario

CCB Scenario: Cnoidal/Cnoidal/Corner/Breaking

Amplitude

Breaking

Corner

Cnoidal
Nonlinear Kelvin Waves on the Sphere
(Boyd and Zhou, *J. Fluid Mech.*, 2009)

- **NO SHEAR: ALL DISPERSION from SPHERICAL GEOMETRY**
- Small amplitude/small $\epsilon$ double perturbation series
- Spectral-Galerkin model
- Newton/continuation
- Kepler change-of-coordinate to resolve the discontinuous slope of the corner wave
- Zoom plots to identify the corner wave as tallest, non-wiggly solution
- Cnoidal/Cornet/Breaking Scenario: All waves above an $\epsilon$-dependent maximum amplitude break, as confirmed by initial-value time-dependent computations.
Kelvin Wave on the Sphere-2

With no mean currents, corner wave occurs at SMALLER and SMALLER amplitude as $\epsilon$ increases.

On the equatorial beta-plane ($\epsilon = \infty$), ALL KELVIN WAVES BREAK

Figure 4: Maximum of $\phi(x, y)$ for the corner wave versus $\epsilon$. The maximum always occurs at the crest of the wave ($X = 0$) and right at the equator ($y = 0$).

$\phi_{00}$ is the height at the peak, $x = 0$ and $y = 0$

$s$=east-west wavenumber
Kelvin Wave on the Sphere-3

As $\epsilon$ decreases, the corner wave profile becomes narrower and narrower in longitude — more soliton-like.

$$\phi(x,y=0) \text{ normalized by } \phi_{00} \ s=2$$

Figure 5: An equatorial cross-section of the height/pressure field $\phi$, $\phi(x, y = 0)$, for several $\epsilon$, normalized by $\phi(0,0)$. 
Kelvin Wave on the Sphere-4

CONE, CREASE or ONE-SIDED POINT SINGULARITY?

Numerical evidence suggests the latter $d\phi/dy$ does NOT have a discontinuity:

![Graph](image)

Figure 6: Longitudinal (solid) and latitudinal (dashed) cross-sections through $\phi(x,y)$ for $s = 2$ and $\epsilon = 30$
Kelvin Wave on the Sphere-5

Two views of the same corner wave below:

\[ s=1 \quad \varepsilon=1 \quad \phi_{00}=0.1905 \]
Kelvin Wave on the Sphere-6

\( d\phi/dx \) has a JUMP DISCONTINUITY at the EQUATOR ONLY
(Insofar as one can judge singularities from numerical computations.)

\( s=2 \ \varepsilon=0.01 \)

Figure 7: \( d\phi/dx \) for \( s = 2 \) and \( \varepsilon = 1/100 \). The right panel is a “zoom” plot of the left panel, showing only 1/15 the range in longitude.
Summary

After 35 years of intermittent studies of the Kelvin wave and one thesis chapter and 14 articles (out of 195) from 1976 to present, still are UNRESOLVED QUESTIONS

• Why does the travelling wave branch, for Kelvin and so many other wave species, terminate in a corner wave?

• Complete classification of generic & non-generic features in corner wave bifurcation.

• How does microbreaking promote mixing in the ocean?

• How does nonlinearity reshape the Kelvin mode’s role in El Niño?

• Does the breaking Kelvin wave overturn or stay single-valued?

• Kelvin corner waves in SHEAR flow
  Critical latitude & resonance issues
“Before I came here I was confused about this subject. Having listened to your lecture I am still confused, but on a higher level.”

Enrico Fermi
Equatorial Beta-Plane: $\epsilon \to \infty$ limit of tidal equations

- Sines & cosines of latitude $\Rightarrow y$ and 1.
- Hough functions become Hermite functions ($v$, or sums of two Hermite functions $(u, \phi)$).
- Tropical is well-approximated, even NONLINEAR, by “one-and-a-half-layer” model
- Linear dynamics is beta-plane form Laplace’s Tidal Equations with actual depth of upper layer (average: 100 m) replaced by equivalent depth (0.4 m).

![Figure 8: Schematic of one-and-a-half layer model. The active layer is described by the shallow water equations.](image-url)
• Kelvin & Rossby waves are disturbances on the “main thermocline”, which is the interface between the warm layer and the cold layer.

• Approximation is not too bad for ocean.

• In one-and-a-half-layer model, wave is only a function of latitude y and longitude x and time t.
Soliton-Machine at Snibston Discovery Park (England)

Recreating solitons in a channel is literally child’s play.
Kelvin Solitary Waves

- Initial-value numerical solutions of shallow water equations confirm solitons in a shear flow.

Figure 10: KELVIN SOLITON: left panels: Initial and final amplitude of the Kelvin mode. Upper right: mean flow & height. Lower right: contours of pressure/height and vector arrows. $u(x, y, 0) = \phi(x, y, 0) = 0.12 \text{sech}^2(0.7x) - \text{constant}.$